Best Constants in the Hardy-Rellich Inequalities and Related Improvements

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Abstract

We consider Hardy-Rellich inequalities and discuss their possible improvement. The procedure is based on decomposition into spherical harmonics, where in addition various new inequalities are obtained (e.g. Rellich-Sobolev inequalities). We discuss also the optimality of these inequalities in the sense that we establish (in most cases) that the constants appearing there are the best ones. Next, we investigate the polyharmonic operator (Rellich and Higher Order Rellich inequalities); the difficulties arising in this case come from the fact that (generally) minimizing sequences are no longer expected to consist of radial functions. Finally, the successively use of the Rellich inequalities lead to various new Higher Order Rellich inequalities.

Keywords: Hardy-Rellich inequalities, Rellich-Sobolev inequalities, Best constants, Optimal inequalities.

1 Introduction

Hardy inequality states that for $N \geq 3$, for all $u \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx. \tag{1.1}$$

The constant $(\frac{N-2}{2})^2$ is the best constant in inequality (1.1). A similar inequality with the same best constant holds if $\mathbb{R}^N$ is replaced by Ω and Ω contains the origin.

When Ω is a bounded domain, a much stronger inequality was discovered by Brezis and Vázquez [BV], that is for all $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2} \right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + z_0^2 \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{2}{N}} \int_\Omega u^2 dx, \tag{1.2}$$

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where ω_N and $|\Omega|$ denote the volume of the unit ball and Ω respectively, and $z_0 = 2.4048\ldots$ denotes the first zero of the Bessel function $J_0(z)$. Inequality (1.2) is optimal in case Ω is a ball centered at zero. We set $D = \sup_{x \in \Omega} |x|$ and define recursively

$$X_1(t) = (1 - \log t)^{-1}, \quad t \in (0, 1],$$
$$X_k(t) = X_1(X_{k-1}(t)), \qquad k = 2, 3, \ldots, t \in (0, 1]. \tag{1.3}$$

In [FT] actually, the following improved Hardy inequality was also established for $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^2} dx + \frac{1}{4} \sum_{i=1}^\infty \int_\Omega \frac{u^2}{|x|^2} X_1^2 X_2^2 \ldots X_i^2 dx, \tag{1.4}$$

where we use the notation X_i for $X_i(\frac{|x|}{D})$. We will make use of the same notation throughout this work. Here the constants that appear are best constants. It is worth mentioning that for $N \geq 2m + 2 > 2$ and $u \in C_0^\infty(\Omega)$ inequality (1.4) takes the equivalent form

$$\int_\Omega \frac{|\nabla u|^2}{|x|^{2m}} dx \geq \left(\frac{N-2m-2}{2}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+2}} dx + \frac{1}{4} \sum_{i=1}^\infty \int_\Omega \frac{u^2}{|x|^{2m+2}} X_1^2 X_2^2 \ldots X_i^2 dx. \tag{1.5}$$

Similarly to (1.1), the classical Rellich inequality states that for $N \geq 5$, for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (\Delta u)^2 \, dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^4} \, dx. \tag{1.6}$$

Davies and Hinz [DH] obtained various Rellich inequalities as well as higher order Rellich inequalities. Gazzola, Grunau and Mitidieri [GGM] on the other hand obtained improved Rellich inequalities in the spirit of [BV]. As an example we mention the following inequality that holds true for $N \geq 5$, and all $u \in C_0^\infty(\Omega)$

$$\int_\Omega (\Delta u)^2 dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4} dx + \frac{N(N-4)}{2} \Lambda_2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{2}{N}} \int_\Omega \frac{u^2}{|x|^2} dx + \Lambda_4^2 \left(\frac{\omega_N}{|\Omega|}\right)^{\frac{4}{N}} \int_\Omega u^2 dx. \tag{1.7}$$

Constants Λ_2, Λ_4 depend only on the space dimension N [GGM].

These type of inequalities arise very naturally in the study of singular differential operators. We would like to mention in particular that improved Hardy inequalities arise in the study of singular solutions of the Gelfand problem [BT, BV], whereas the improved Rellich in the biharmonic analogue of the Gelfand problem [GGM]. It is worth noting the work of Eilertsen [E] which is connected with the work of Maz'ya [M2] on the Wiener test for higher order Elliptic equations. Related are also the works of Yafaev [Y] and Grillo [GG]. For some recent results concerning Hardy-Sobolev inequalities we refer to [A, ACR, HN, MS, V].

Our aim in this paper is to obtain sharp improved versions of inequalities such as (1.6) and (1.7), where additional non-negative terms are present in the respective right-hand sides. At the same time we obtain some new improved Rellich inequalities which are new even at the level of plain Rellich inequalities. The method we use was first introduced in [FT] to obtain Hardy inequalities, here we extend it to obtain higher order Rellich inequalities. Attached to the Rellich inequality (1.6), there is a similar Rellich inequality that connects first to second order derivatives. That is, for $N \geq 5$, and for all $u \in \mathbb{R}^N$ we have

$$\int_{\mathbb{R}^N} (\Delta u)^2 \, dx \geq \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2} \, dx. \tag{1.8}$$

The constant $\frac{N^2}{4}$ is the best constant for (1.8). From this inequality and from (1.5) we easily arrive to a much stronger inequality than (1.6). It was a surprise for us that we have not trace inequality (1.8) in the literature.

From now on Ω is a bounded domain containing the origin. In Ω inequalities (1.6) and (1.8) take the following much stronger form.

Theorem 1.1 (Improved Rellich-Sobolev inequality) *Let $N \geq 5$ and $D \geq \sup_{x \in \Omega} |x|$. There exists a positive constant c such that for all $u \in H_0^2(\Omega)$ there holds*

$$(ii) \qquad \int_\Omega (\Delta u)^2 dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx + c \left(\int_\Omega |u|^{\frac{2N}{N-4}} X_1^{\frac{2(N-2)}{N-4}} \, dx\right)^{\frac{N-4}{N}}. \qquad (1.9)$$

$$(i) \qquad \int_\Omega (\Delta u)^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + c \left(\int_\Omega |\nabla u|^{\frac{2N}{N-2}} X_1^{\frac{2(N-1)}{N-2}} \, dx\right)^{\frac{N-2}{N}}, \qquad (1.10)$$

Let us now give the following

Definition 1.2 *(Optimal Inequality) Suppose that for some potential V, we have for all $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega (\Delta u)^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + \int_\Omega |\nabla u|^2 \, V \, dx. \qquad (1.11)$$

We say that inequality (1.11) is **optimal***, when there is no potential $W \gneqq 0$ to make the inequality*

$$\int_\Omega (\Delta u)^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + \int_\Omega |\nabla u|^2 \, V \, dx + \int_\Omega |\nabla u|^2 \, W \, dx, \qquad (1.12)$$

hold true for all $u \in C_0^\infty(\Omega)$.

We then have

Theorem 1.3 (Improved Rellich inequality I) *Let $N \geq 5$ and $D \geq \sup_{x \in \Omega} |x|$.*
(i) Suppose the potential $V \gneqq 0$ is such that

$$\int_\Omega V^{\frac{N}{2}} X_1^{1-N} \, dx \quad < \quad +\infty. \qquad (1.13)$$

Then there exists a positive constant b such that for all $u \in C_0^\infty(\Omega)$, there holds

$$\int_\Omega (\Delta u)^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + b \int_\Omega |\nabla u|^2 \, V \, dx. \qquad (1.14)$$

If in addition b is the best constant, then inequality (1.14) is an optimal inequality.
(ii) Suppose the potential $W \gneqq 0$ is such that

$$\int_\Omega W^{\frac{N}{4}} X_1^{1-\frac{N}{2}} \, dx \quad < \quad +\infty. \qquad (1.15)$$

Then there exists a positive constant c such that for all $u \in C_0^\infty(\Omega)$, there holds

$$\int_\Omega (\Delta u)^2 dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{|u|^2}{|x|^4} \, dx + c \int_\Omega |u|^2 \, V \, dx. \qquad (1.16)$$

If in addition c is the best constant, then inequality (1.16) is an optimal inequality.

The difficult part in the previous theorem is establishing that inequalities are optimal and we will do that in section 4. We can improve Rellich inequality differently and obtain

Theorem 1.4 (Improved Rellich inequality II) *Let $N \geq 5$ and $D \geq \sup_{x \in \Omega} |x|$. Then for all $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega (\Delta u)^2 \, dx \geq \left(\frac{N(N-4)}{4} \right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx + \left(1 + \frac{N(N-4)}{8} \right) \sum_{i=1}^\infty \int_\Omega \frac{u^2}{|x|^4} X_1^2 X_2^2 \dots X_i^2 \, dx. \quad (1.17)$$

Moreover, for each $k = 1, 2, \dots$, the constant $\left(1 + \frac{N(N-4)}{8} \right)$ is the best constant for the corresponding k-Improved Rellich Inequality, that is

$$1 + \frac{N(N-4)}{8} =$$

$$\inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega (\Delta u)^2 \, dx - \left(\frac{N(N-4)}{4} \right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx - \left(1 + \frac{N(N-4)}{8} \right) \sum_{i=1}^{k-1} \int_\Omega \frac{u^2}{|x|^4} X_1^2 X_2^2 \dots X_i^2 \, dx}{\int_\Omega \frac{u^2}{|x|^4} X_1^2 X_2^2 \dots X_k^2 \, dx}. \quad (1.18)$$

Theorem 1.5 (Improved Rellich inequality III) *Let $N \geq 5$ and $D \geq \sup_{x \in \Omega} |x|$. Then for all $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega (\Delta u)^2 \, dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + \frac{1}{4} \sum_{i=1}^\infty \int_\Omega \frac{|\nabla u|^2}{|x|^2} X_1^2 X_2^2 \dots X_i^2 \, dx. \quad (1.19)$$

Moreover, the constant $\frac{N^2}{4}$ is the best and similarly for each $k = 1, 2, \dots$, the constant $\frac{1}{4}$ is the best constant for the corresponding k-Improved Rellich Inequality, that is

$$\frac{1}{4} = \inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega (\Delta u)^2 \, dx - \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx - \frac{1}{4} \sum_{i=1}^{k-1} \int_\Omega \frac{|\nabla u|^2}{|x|^2} X_1^2 X_2^2 \dots X_i^2 \, dx}{\int_\Omega \frac{|\nabla u|^2}{|x|^2} X_1^2 X_2^2 \dots X_k^2 \, dx}. \quad (1.20)$$

Next we consider higher order Rellich inequalities. When applying Theorems 1.5, 1.4 we reduce the order by one or two. In doing so weights enter in our inequalities. For this reason we first consider second order Rellich inequalities with weights. For $N \geq 5$ and $0 \leq m < \frac{N-4}{2}$, holds that

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} \, dx \geq \left(\frac{(N+2m)(N-4-2m)}{4} \right)^2 \int_\Omega \frac{u^2}{|x|^{2m+4}} \, dx, \quad (1.21)$$

while the corresponding improved inequalities can be stated as

Theorem 1.6 (Improved Rellich inequality IV) *Suppose $N \geq 5$, $0 \leq m < \frac{N-4}{2}$ and $D \geq \sup_{x \in \Omega} |x|$. Then for all $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega \frac{(\Delta u)^2}{|x|^{2m}} \, dx \geq \left(\frac{(N+2m)(N-4-2m)}{4} \right)^2 \int_\Omega \frac{u^2}{|x|^{2m+4}} \, dx$$

$$+ \left((1+m)^2 + \frac{(N+2m)(N-4-2m)}{8} \right) \sum_{i=1}^\infty \int_\Omega \frac{u^2}{|x|^{2m+4}} X_1^2 X_2^2 \dots X_i^2 \, dx. \quad (1.22)$$

Moreover $(\frac{(N+2m)(N-4-2m)}{4})^2$ is the best constant. Similarly for each $k = 1, 2, \ldots$, the constant $(1+m)^2 +$ $\frac{(N+2m)(N-4-2m)}{8}$ is the best constant for the corresponding k-Improved Hardy-Rellich Inequality, that is

$$(1+m)^2 + \frac{(N+2m)(N-4-2m)}{8} =$$

$$\inf_{u \in C_0^\infty(\Omega)} \frac{\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} dx - \left(\frac{(N+2m)(N-4-2m)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+4}} dx - A \sum_{i=1}^{k-1} \int_\Omega \frac{u^2}{|x|^{2m+4}} X_1^2 \ldots X_i^2 \, u^2 dx}{\int_\Omega \frac{u^2}{|x|^{2m+4}} X_1^2 \ldots X_k^2 \, dx}, \quad (1.23)$$

where $A = (1+m)^2 + \frac{(N+2m)(N-4-2m)}{8}$.

On the other hand the weighted Rellich inequality of the form (1.8) reads:

Theorem 1.7 *Suppose $N \geq 5$ and $0 \leq m < \frac{N-4}{2}$. Then, for all $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} dx \geq a_{m,N} \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} dx, \quad (1.24)$$

where the best constant $a_{m,N}$ is given by:

$$a_{m,N} := \min_{k=0,1,2,\ldots} \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + k(N+k-2)\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + k(N+k-2)}. \quad (1.25)$$

In particular when $0 \leq m \leq \frac{-(N+4)+2\sqrt{N^2-N+1}}{6}$, we have

$$a_{m,N} = \left(\frac{N+2m}{2}\right)^2,$$

whereas when $\frac{-(N+4)+2\sqrt{N^2-N+1}}{6} < m < \frac{N-4}{2}$, we have

$$0 < a_{m,N} < \left(\frac{N+2m}{2}\right)^2.$$

In Theorem 6.6 we have a full description of how the constant $a_{m,N}$ behaves. Our next result is

Theorem 1.8 (Improved Rellich inequality V) *Let $D \geq \sup_{x \in \Omega} |x|$ and $0 \leq m \leq \frac{-(N+4)+2\sqrt{N^2-N+1}}{6}$. Then for all $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} dx - \left(\frac{N+2m}{2}\right)^2 \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} dx \geq \frac{1}{4} \sum_{i=1}^\infty \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} X_1^2 X_2^2 \ldots X_i^2 \, dx. \quad (1.26)$$

Moreover for each $k = 1, 2, \ldots$, the constant $\frac{1}{4}$ is the best constant for the corresponding k-Improved Hardy-Rellich Inequality, that is

$$\frac{1}{4} = \inf_{u \in H_0^2(\Omega)} \frac{\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} dx - \left(\frac{N+2m}{2}\right)^2 \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} dx - \frac{1}{4} \sum_{i=1}^{k-1} \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} X_1^2 X_2^2 \ldots X_i^2 \, dx}{\int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} X_1^2 X_2^2 \ldots X_k^2 \, dx}. \quad (1.27)$$

In order to state our improved higher order Rellich inequality we set

$$\sigma(m, N) = \left(\frac{(N + 2m)(N - 4 - 2m)}{4} \right)^2 \tag{1.28}$$

$$\bar{\sigma}(m, N) = (1 + m)^2 + \frac{(N + 2m)(N - 4 - 2m)}{8}. \tag{1.29}$$

We then have

Theorem 1.9 (Improved Higher Order Rellich Inequalities I) *Suppose* $m \in \mathbb{N}$, $l = 0, \cdots, m - 1$, $4m < N$ *and* $D \geq \sup_{x \in \Omega} |x|$. *Then for all* $u \in C_0^\infty(\Omega)$ *there holds*

(i) $\displaystyle \int_\Omega (\Delta^m u)^2 \, dx \;\geq\; \prod_{k=0}^{l} \left(\frac{(N + 4k)(N - 4 - 4k)}{4} \right)^2 \int_\Omega \frac{(\Delta^{m-l-1} u)^2}{|x|^{4l+4}} \, dx$

$$+ \; \sum_{k=1}^{l} \bar{\sigma}(2k, N) \prod_{j=0}^{k-1} \sigma(2j, N) \sum_{i=1}^{\infty} \int_\Omega \frac{(\Delta^{m-k-1} u)^2}{|x|^{4k+4}} X_1^2 \dots X_i^2 dx$$

$$+ \; \left(1 + \frac{N(N-4)}{8} \right) \sum_{i=1}^{\infty} \int_\Omega \frac{(\Delta^{m-1} u)^2}{|x|^4} X_1^2 \dots X_i^2 \, dx \;, \tag{1.30}$$

(ii) $\displaystyle \int_\Omega |\nabla \Delta^m u|^2 \, dx \;\geq\; \left(\frac{N-2}{2} \right)^2 \prod_{k=0}^{l-1} \left(\frac{(N + 2 + 4k)(N - 6 - 4k)}{4} \right)^2 \int_\Omega \frac{(\Delta^{m-l} u)^2}{|x|^{4l+2}} \, dx$

$$+ \; \left(\frac{N-2}{2} \right)^2 \sum_{k=2}^{l} \bar{\sigma}(2k-1, N) \prod_{j=0}^{k-2} \sigma(2j+1, N) \sum_{i=1}^{\infty} \int_\Omega \frac{(\Delta^{m-k} u)^2}{|x|^{4k+2}} X_1^2 \dots X_i^2 \, dx$$

$$+ \; \left(\frac{N-2}{2} \right)^2 \left(4 + \frac{(N+2)(N-6)}{8} \right) \sum_{i=1}^{\infty} \int_\Omega \frac{(\Delta^{m-1} u)^2}{|x|^6} X_1^2 \dots X_i^2 \, dx$$

$$+ \; \frac{1}{4} \sum_{i=1}^{\infty} \int_\Omega \frac{(\Delta^m u)^2}{|x|^2} X_1^2 \dots X_i^2 \, dx \;. \tag{1.31}$$

Theorem 1.10 (Improved Higher Order Rellich Inequality II) *Suppose* $m, l \in \mathbb{N}$, $1 \leq l \leq \frac{-N + 8 + 2\sqrt{N^2 - N + 1}}{12}$, $4m < N$ *and* $D \geq \sup_{x \in \Omega} |x|$. *Then for all* $u \in C_0^\infty(\Omega)$ *there holds*

$$\int_\Omega (\Delta^m u)^2 \, dx \;\geq\; \prod_{k=0}^{l-1} \left(\frac{(N + 4k)(N - 4 - 4k)}{4} \right)^2 \int_\Omega \frac{(\Delta^{m-l} u)^2}{|x|^{4l}} \, dx$$

$$+ \; \frac{4}{(N-4)^2} \sum_{k=1}^{l} \prod_{j=0}^{k-1} \left(\frac{(N - 4 + 4j)(N - 4j)}{4} \right)^2 \sum_{i=1}^{\infty} \int_\Omega \frac{(\nabla \Delta^{m-k} u)^2}{|x|^{4k-2}} X_1^2 \dots X_i^2 \, dx$$

$$+ \; \frac{1}{N^2} \sum_{k=1}^{l} \prod_{j=1}^{k} \left(\frac{(N + 4j)(N - 4j)}{4} \right)^2 \sum_{i=1}^{\infty} \int_\Omega \frac{(\Delta^{m-k} u)^2}{|x|^{4k}} X_1^2 \dots X_i^2 \, dx \;. \tag{1.32}$$

The paper is divided in two parts. In the first part we deal with the biharmonic operator, while in the second part we deal with the polyharmonic operator. More precisely, in Section 2 we prove some identities and inequalities to be used widely in the sequel; the main tool for this is decomposition into spherical harmonics. In Section 3 we prove Theorems 1.1, 1.4 and 1.5, while in Section 4 we prove that

the constants appearing in certain inequalities are the best and complete the proof of Theorems 1.4 and 1.5. In Section 5 we state necessary conditions for the improvement or not of inequalities (1.6), (1.8). In the two Sections of Part II we actually prove Theorems 1.6 to 1.10.

Notation: Sometimes, for the sake of the representation we use the following quantities

$$
\begin{aligned}
I_\Omega[u] &:= \int_\Omega |\Delta u|^2 \, dx - \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx, \\
J_\Omega[v] &:= \int_\Omega |x|^{-(N-4)} |\Delta v|^2 \, dx - N(N-4) \int_\Omega |x|^{-N}(x \cdot \nabla v)^2 \, dx \\
&\quad + \frac{N(N-4)}{2} \int_\Omega |x|^{-(N-2)} |\nabla v|^2 \, dx, \\
\mathbb{I}_\Omega[u] &:= \int_\Omega |\Delta u|^2 \, dx - \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx, \\
\mathbb{J}_\Omega[v] &:= \int_\Omega |x|^{-(N-4)} |\Delta v|^2 \, dx - N(N-4) \int_\Omega |x|^{-N}(x \cdot \nabla v)^2 \, dx \\
&\quad + \frac{N(N-8)}{4} \int_\Omega |x|^{-(N-2)} |\nabla v|^2 \, dx,
\end{aligned}
$$

related to (1.6) and (1.8).

PART I. THE BIHARMONIC OPERATOR

2 Preliminaries

In this section we establish some abstract relations to be used in the sequel. In the first part we prove some useful identities while, in the second part we apply spherical harmonic decomposition in order to prove certain inequalities. Throughout this section Ω is an arbitrary domain (bounded or unbounded).

2.1 Preliminaries Identities

Lemma 2.1 *Let $N \geq 3$, $a < N-2$ and $B \in C^2[0, +\infty)$. Then, for any $u \in C_0^\infty(\Omega)$, we have the identity:*

$$
\int_\Omega \frac{B(r)}{r^a} |\nabla u|^2 \, dx = - \int_\Omega \frac{B(r)}{r^a} u \, \Delta u \, dx + \frac{1}{2} \int_\Omega \Delta\left(\frac{B(r)}{r^a}\right) u^2 \, dx.
$$

Proof Observe that for $a < N-2$ we have that $\Delta\left(\frac{B(r)}{r^a}\right) u^2 \in L^1(\Omega)$ and $\nabla\left(\frac{B(r)}{r^a}\right) \nabla u^2 \in L^1(\Omega)$. In virtue of the identity

$$
|\nabla w|^2 = \frac{1}{2} \Delta w^2 - w \Delta w, \tag{2.1}
$$

it suffices to prove that

$$
\int_\Omega \frac{B(r)}{r^a} \Delta u^2 \, dx = \int_\Omega \Delta\left(\frac{B(r)}{r^a}\right) u^2 \, dx. \tag{2.2}
$$

If we write

$$
\int_\Omega \frac{B(r)}{r^a} \Delta u^2 \, dx = \int_{\Omega \setminus B_\varepsilon} \frac{B(r)}{r^a} \Delta u^2 \, dx + \int_{B_\varepsilon} \frac{B(r)}{r^a} \Delta u^2 \, dx,
$$

and using the limits

$$\left| \int_{\partial B_\varepsilon} \frac{\partial}{\partial \nu} \left(\frac{B(r)}{r^a} \right) u^2 \, ds \right| \le c \, \varepsilon^{N-2-a} \to 0 \quad \text{and} \quad \left| \int_{\partial B_\varepsilon} \frac{B(r)}{r^a} \frac{\partial}{\partial \nu} u^2 \, ds \right| \le c \, \varepsilon^{N-1-a} \to 0,$$

as $\varepsilon \to 0$, we obtain that (2.2) is true. Thus, the proof is completed. ∎

Lemma 2.2 *Let $N > 4$ and $0 < a \le \frac{N-4}{2}$. For any $u \in C_0^\infty(\Omega)$ we set $v = |x|^a u$. Then, the following equality holds.*

$$\int_\Omega |\Delta u|^2 \, dx = \int_\Omega |x|^{-2a} |\Delta v|^2 \, dx - 4a(a+2) \int_\Omega |x|^{-2a-4} (x \cdot \nabla v)^2 \, dx$$

$$+ 2a(a+2) \int_\Omega |x|^{-2a-2} |\nabla v|^2 \, dx$$

$$+ a(a+2)(-N+a+2)(-N+a+4) \int_\Omega |x|^{-2a-4} v^2 \, dx, \tag{2.3}$$

Proof We have that

$$\int_\Omega |\Delta u|^2 \, dx = \int_\Omega |x|^{-2a} |\Delta v|^2 \, dx + 4a^2 \int_\Omega |x|^{-2a-4} (x \cdot \nabla v)^2 \, dx$$

$$+ \left(-aN + a(a+2) \right)^2 \int_\Omega |x|^{-2a-4} v^2 \, dx + I_1 + I_2 + I_3, \tag{2.4}$$

where

$$I_1 := 2 \int_\Omega |x|^{-a} \Delta v \, v \Delta |x|^{-a} \, dx$$

$$I_2 := 4 \int_\Omega |x|^{-a} \Delta v \, \nabla v \cdot \nabla |x|^{-a} \, dx$$

$$I_3 := 4 \int_\Omega \nabla |x|^{-a} \cdot \nabla v \, v \Delta |x|^{-a} \, dx.$$

Following the same procedure as in the proof of Lemma 2.1 (having in mind also that $|v| \le |x|^a ||u||_\infty$) we obtain that

$$I_1 = a(-N+a+2)(2a+2)(-N+2a+4) \int_\Omega |x|^{-2a-4} v^2 \, dx$$

$$- 2 \left(-aN + a(a+2) \right) \int_\Omega |x|^{-2a-2} |\nabla v|^2 \, dx,$$

$$I_2 = 4a(-2a-2) \int_\Omega |x|^{-2a-4} (x \cdot \nabla v)^2 \, dx + 2a(-N+2a+4) \int_\Omega |x|^{-2a-2} |\nabla v|^2 \, dx,$$

$$I_3 = 2a^2(-N+a+2)(-N+2a+4) \int_\Omega |x|^{-2a-4} v^2 \, dx.$$

Then, from (2.4) we conclude (2.3) and the proof is completed. ∎

Using the previous lemma we may easily obtain the following result, concerning the relation between $I, \mathbb{I}, J, \mathbb{J}$.

Lemma 2.3 *Let $N \ge 5$, $u \in C_0^\infty(\Omega)$ and $v = |x|^{(N-4)/2} u$. We have that:*

$$i) \quad \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx = \int_\Omega |x|^{-(N-2)} |\nabla v|^2 \, dx + \left(\frac{N-4}{2} \right)^2 \int_\Omega |x|^{-N} |v|^2 \, dx,$$

$$ii) \quad \int_\Omega |\Delta u|^2\, dx - \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4}\, dx = \int_\Omega |x|^{-(N-4)}|\Delta v|^2\, dx -$$

$$- N(N-4)\int_\Omega |x|^{-N}(x\cdot\nabla v)^2\, dx + \frac{N(N-4)}{2}\int_\Omega |x|^{-(N-2)}|\nabla v|^2\, dx,$$

$$iii) \quad \int_\Omega |\Delta u|^2\, dx - \frac{N^2}{4}\int_\Omega \frac{|\nabla u|^2}{|x|^2}\, dx = \int_\Omega |x|^{-(N-4)}|\Delta v|^2\, dx -$$

$$- N(N-4)\int_\Omega |x|^{-N}(x\cdot\nabla v)^2\, dx + \frac{N(N-8)}{4}\int_\Omega |x|^{-(N-2)}|\nabla v|^2\, dx,$$

2.2 Preliminaries Inequalities

The decomposition of u and v into spherical harmonics will be one of the main tools in our investigation. Let $u \in C_0^\infty(\Omega)$. If we extend u as zero outside Ω, we may consider that $u \in C_0^\infty(\mathbb{R}^N)$. Decomposing u into spherical harmonics we get

$$u = \sum_{k=0}^\infty u_k := \sum_{k=0}^\infty f_k(r)\phi_k(\sigma),$$

where $\phi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k = k(N+k-2)$, $k \geq 0$. The functions f_k belong in $C_0^\infty(\Omega)$, satisfying $f_k(r) = O(r^k)$ and $f_k'(r) = O(r^{k-1})$, as $r \downarrow 0$. In particular, $\phi_0(\sigma) = 1$ and $u_0(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} u\, ds$, for any $r > 0$. Then, for any $k \in \mathbb{N}$, we have that

$$\Delta u_k = \left(\Delta f_k(r) - \frac{c_k f_k(r)}{r^2}\right)\phi_k(\sigma)$$

so

$$\int_{\mathbb{R}^N} |\Delta u_k|^2\, dx = \int_{\mathbb{R}^N} \left(\Delta f_k(r) - \frac{c_k f_k(r)}{r^2}\right)^2 dx. \tag{2.5}$$

In addition,

$$\int_{\mathbb{R}^N} |\nabla u_k|^2\, dx = \int_{\mathbb{R}^N} \left(|\nabla f_k(r)|^2 + c_k \frac{f_k^2(r)}{r^2}\right) dx. \tag{2.6}$$

Next, we assume the function $v \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$, such that $v = |x|^{\frac{N-4}{2}}u$. From the definitions of u and v, we may write that

$$u = \sum_{k=0}^\infty u_k = \sum_{k=0}^\infty r^{\frac{-N+4}{2}+k}g_k(r)\phi_k(\sigma), \quad v = \sum_{k=0}^\infty v_k = \sum_{k=0}^\infty r^k g_k(r)\phi_k(\sigma),$$

where $f_k = r^{-\frac{N-4}{2}+k}g_k$, with $g_k \sim 0$ and $r\, g_k' \sim 0$ at the origin. More precisely, we may prove that the following identities hold, for any $k \in \mathbb{N}$.

$$\int_{\mathbb{R}^N} |\Delta u_k|^2 dx = \int_{\mathbb{R}^N} r^{2k-N+4}|\nabla g_k'|^2 dx + \left(\frac{N(N-4)}{2} + 2k(N-3) + 3\right)\int_{\mathbb{R}^N} r^{2k-N+2}|\nabla g_k|^2 dx$$

$$+ \left[\left(\frac{N(N-4)}{4}\right)^2 + \frac{N(N-4)}{2}(c_k + k^2)\right]\int_{\mathbb{R}^N} r^{2k-N}(g_k)^2\, dx \tag{2.7}$$

$$\int_{\mathbb{R}^N} |x|^{-2}|\nabla u_k|^2\, dx = \int_{\mathbb{R}^N} r^{2k-N+2}|\nabla g_k|^2\, dx + \left[\left(\frac{N-4}{2}\right)^2 + k(N-2)\right]\int_{\mathbb{R}^N} r^{2k-N}(g_k)^2\, dx, \tag{2.8}$$

$$I[u_k] = \int_{\mathbb{R}^N} r^{2k-N+4} |\nabla g_k'|^2 \, dx + \left(\frac{N(N-4)}{2} + 2k(N-3) + 3 \right) \int_{\mathbb{R}^N} r^{2k-N+2} |\nabla g_k|^2 \, dx$$
$$+ \left[\frac{N(N-4)}{2} (c_k + k^2) \right] \int_{\mathbb{R}^N} r^{2k-N} (g_k)^2 \, dx, \tag{2.9}$$

$$\mathbb{I}[u_k] = \int_{\mathbb{R}^N} r^{2k-N+4} |\nabla g_k'|^2 \, dx + \left((2k+N-1)(N-3) - \frac{N(3N-8)}{4} \right) \int_{\mathbb{R}^N} r^{2k-N+2} |\nabla g_k|^2 \, dx$$
$$+ \left[\frac{N(3N-8)}{4} k^2 + \frac{N(N-8)}{4} c_k \right] \int_{\mathbb{R}^N} r^{2k-N} (g_k)^2 \, dx, \tag{2.10}$$

$$\int_{\mathbb{R}^N} r^{-(N-4)} |\Delta v_k|^2 \, dx = \int_{\mathbb{R}^N} r^{2k-N+4} |\nabla g_k'|^2 \, dx + (2k+N-1)(N-3) \int_{\mathbb{R}^N} r^{2k-N+2} |\nabla g_k|^2 \, dx, \tag{2.11}$$

$$\int_{\mathbb{R}^N} r^{-(N-2)} |\nabla v_k|^2 \, dx = \int_{\mathbb{R}^N} r^{2k-N+2} |\nabla g_k|^2 \, dx + k(N-2) \int_{\mathbb{R}^N} r^{2k-N} (g_k)^2 \, dx, \tag{2.12}$$

$$\int_{\mathbb{R}^N} r^{-N} (x \cdot \nabla v_k)^2 \, dx = \int_{\mathbb{R}^N} r^{2k-N+2} |\nabla g_k|^2 \, dx - k^2 \int_{\mathbb{R}^N} r^{2k-N} (g_k)^2 \, dx. \tag{2.13}$$

Let $k = 1, 2, ...$ and $V(|x|) \in C^1([0, +\infty))$. The following relation

$$\int_{\mathbb{R}^N} V(|x|) \, |x|^{-2} |\nabla u_k|^2 dx = \int_{\mathbb{R}^N} V(|x|) \, |x|^{-2} |\nabla f_k|^2 dx + c_k \int_{\mathbb{R}^N} V(|x|) \, |x|^{-4} f_k^2 dx.$$

implies that

$$\int_{\mathbb{R}^N} V(|x|) \, |x|^{-2} |\nabla u_k|^2 \, dx = \int_{\mathbb{R}^N} r^{2k+2-N} V(|x|) \, |\nabla g_k|^2 \, dx +$$
$$+ \left[\left(\frac{N-4}{2} \right)^2 + k(N-2) \right] \int_{\mathbb{R}^N} r^{2k-N} V(|x|) \, g_k^2 \, dx + \left(\frac{N-4}{2} - k \right) \int_{\mathbb{R}^N} r^{2k+1-N} V'(|x|) \, g_k^2 \, dx. \tag{2.14}$$

Also, as an immediate consequence of the Hardy inequality, we have the following relations.

$$\int_0^\infty r^{2k+3} (g_k'')^2 \, dr \;\; \geq \;\; (k+1)^2 \int_0^\infty r^{2k+1} (g_k')^2 \, dr, \tag{2.15}$$

$$\int_0^\infty r^{2k+1} (g_k')^2 \, dr \;\; \geq \;\; k^2 \int_0^\infty r^{2k-1} g_k^2 \, dr. \tag{2.16}$$

Observe that in the case of a bounded domain Ω, all the obtained equalities remain true if we assume B_D, with $D = sup_{x \in \Omega} |x|$, instead of $\mathbb{R}^N$.

In the remaining part of this section, using the decomposition into spherical harmonics, we establish certain inequalities concerning $I[u]$ and $\mathbb{I}[u]$.

Theorem 2.4 *Let $N \geq 5$, $u \in C_0^\infty(\Omega)$ and $v = |x|^{(N-4)/2} u$. Then*

$$i) \quad \int_\Omega |\Delta u|^2 \, dx - \left(\frac{N(N-4)}{4} \right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx \geq \left(4 + \frac{N(N-4)}{2} \right) \int_\Omega |x|^{-(N-2)} |\nabla v|^2 \, dx. \tag{2.17}$$

$$ii) \quad \int_\Omega |\Delta u|^2 \, dx - \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx \geq \left(\frac{N-4}{2} \right)^2 \int_\Omega |x|^{-(N-2)} |\nabla v|^2 \, dx. \tag{2.18}$$

Proof i) It suffices to prove, by using (2.9), (2.12) and (2.15), that the following inequality

$$\left[(k+N-2)^2 - \frac{N(N-4)}{2} - \left(4 + \frac{N(N-4)}{2}\right)\right]\int_0^\infty r^{2k+1}(g_k')^2 \, dr \geq$$
$$\left[k(N-2)\left(4 + \frac{N(N-4)}{2}\right) - \frac{N(N-4)}{2}(k^2 + c_k)\right]\int_0^\infty r^{2k-1}(g_k)^2 \, dr, \qquad (2.19)$$

holds for any $k = 1, 2, ...$ or equivalently

$$\left(k + 2N - 4\right)\int_0^\infty r^{2k+1}(g_k')^2 \, dr \geq \left[4(N-2) - kN(N-4)\right]\int_0^\infty r^{2k-1}(g_k)^2 \, dr,$$

which is true since

$$k^2 \geq \frac{4(N-2) - kN(N-4)}{k + 2N - 4},$$

for $k = 1, 2, ...$ and $N \geq 5$.

ii) From Lemma 2.3 we deduce that

$$\mathbb{I}_\Omega[u] = I_\Omega[u] - \frac{N^2}{4}\int_\Omega |x|^{-(N-2)}|\nabla v|^2 \, dx.$$

Then, the result follows from (2.17). ∎

Lemma 2.5 *Let* $N \geq 5$, $u \in C_0^\infty(\Omega)$ *and* $v = |x|^{(N-4)/2}u$. *Then, the following inequalities hold.*

$$\int_\Omega |x|^{4-N}|\Delta v|^2 \, dx \geq N(N-4)\int_\Omega |x|^{-N}(x \cdot \nabla v)^2 \, dx + 4\int_\Omega |x|^{2-N}|\nabla v|^2 \, dx \qquad (2.20)$$

$$\int_\Omega |x|^{4-N}|\Delta v|^2 \, dx \geq 2(N-2)^2\left(\int_\Omega |x|^{-N}(x \cdot \nabla v)^2 \, dx - \frac{1}{2}\int_\Omega |x|^{2-N}|\nabla v|^2 \, dx\right). \qquad (2.21)$$

Proof Inequality (2.20) follows Theorem 2.4, while (2.21) follows from (2.20) and the following inequality

$$\int_\Omega |x|^{-N}|x \cdot \nabla v|^2 \, dx - \frac{1}{2}\int_\Omega |x|^{2-N}|\nabla v|^2 \, dx \leq$$
$$\frac{1}{2(N-2)^2}\left[N(N-4)\int_\Omega |x|^{-N}|x \cdot \nabla v|^2 \, dx + 4\int_\Omega |x|^{2-N}|\nabla v|^2 \, dx\right]. \quad ∎ \qquad (2.22)$$

An immediate consequence of the inequality (2.21) is the following result.

Corollary 2.6 *Let* $N \geq 5$, $u \in C_0^\infty(\Omega)$ *and* $v = |x|^{(N-4)/2}u$. *Then*

$$\int_\Omega |\Delta u|^2 \, dx - \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx \geq \left(\frac{1}{2} + \frac{2}{(N-2)^2}\right)\int_\Omega |x|^{-(N-4)}|\Delta v|^2 \, dx. \qquad (2.23)$$

Theorem 2.7 *Let* $N \geq 5$, $u \in C_0^\infty(\Omega)$ *and* $v = |x|^{(N-4)/2}u$. *Then*

$$\int_\Omega |\Delta u|^2 \, dx - \frac{N^2}{4}\int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx \geq \left(\frac{N-4}{2(N-2)}\right)^2 \int_\Omega |x|^{-(N-4)}|\Delta v|^2 \, dx. \qquad (2.24)$$

Proof Using the identities (2.10) and (2.11) we have that (2.24) holds if the following inequality

$$A := \int_0^\infty r^{2k+3}(g_k'')^2\, dr + (2kN - 6k - 1)\int_0^\infty r^{2k+1}(g_k')^2\, dr$$

$$+ (N-2)^2\left(k^2 + \frac{N-8}{3N-8}c_k\right)\int_0^\infty r^{2k-1}g_k^2\, dr \geq 0,$$

is true for any $k \in \mathbb{N}$. Taking now into account (2.15) and (2.16) we deduce that for any $k \in \mathbb{N}$ holds that

$$A \geq A(k)\int_0^\infty r^{2k-1}g_k^2\, dr,$$

where

$$A(k) = k^2(k^2 + 2kN - 4k) + k(N-2)^2\left(k + \frac{N-8}{3N-8}(k+N-2)\right).$$

It is clear that $A(k)$ is an increasing function for positive k, with $A(0) = 0$. Thus, $A \geq 0$, for any $k \in \mathbb{N}$ and the proof is completed. $\blacksquare$

3   Hardy-Sobolev and Improved Inequalities

In this section we prove certain Hardy-Sobolev-type inequalities and we establish some improved Hardy inequalities. Throughout this section we assume that $N \geq 5$, Ω is a bounded domain and $D = sup_{x\in\Omega}|x|$. We extend any $u \in C_0^\infty(\Omega)$ as zero outside Ω so we consider that $u \in C_0^\infty(\mathbb{R}^N)$. We then define $u_0(r) := \frac{1}{|\partial B_r|}\int_{\partial B_r} u\, ds$, for any $r > 0$. It is clear that $u_0 \in C_0^\infty[0, D)$.

Theorem 3.1 *Let Ω be a bounded domain, $D = sup_{x\in\Omega}|x|$ and $u \in C_0^\infty(\Omega)$. Then,*

$$I_\Omega[u] \geq I_{B_D}[u_0] + \frac{8(N-1)(N^2 - 2N - 2)}{(N^2-4)^2}\int_{B_D} |\Delta(u - u_0)|^2\, dx. \tag{3.1}$$

Proof Observe that $I_\Omega[u] = I_{B_D}[u_0] + \sum_{k=1}^\infty I_{B_D}[u_k]$. It suffices to prove that for any $k = 1, \ldots$, holds that

$$I_{B_D}[u_k] \geq \frac{8(N-1)(N^2 - 2N - 2)}{(N^2-4)^2}\int_{B_D} |\Delta u_k|^2\, dx.$$

Assume that the following inequality holds

$$I_{B_D}[u_k] \geq a\int_{B_D} |\Delta u_k|^2\, dx,$$

for some $0 < a < 1$ and any $k = 1, 2, \ldots$. Taking into account (2.7) and (2.9) we obtain that

$$\int_{B_D} r^{2k-N+4}|\nabla g'|^2\, dx + \left(3 + 2k(N-3) + \frac{N(N-4)}{2}\right)\int_{B_D} r^{2k-N+2}|\nabla g|^2\, dx \geq$$

$$\geq \frac{1}{1-a}\left\{a\left[\left(\frac{N(N-4)}{4}\right)^2 + \frac{N(N-4)}{2}(c_k + k^2)\right] - \frac{N(N-4)}{2}(c_k + k^2)\right\}\int_{B_D} r^{2k-N}g^2\, dx.$$

Using now (2.15) and (2.16) we deduce that $a \leq G(k)$, where

$$G(k) = \frac{k^2\left(3 + 2k(N-3) + (k+1)^2 + \frac{N(N-4)}{2}\right) + \frac{N(N-4)}{2}\left(2k^2 + k(N-2)\right)}{\left(\frac{N(N-4)}{4}\right)^2 + k^2\left(3 + 2k(N-3) + (k+1)^2 + \frac{N(N-4)}{2}\right) + \frac{N(N-4)}{2}\left(2k^2 + k(N-2)\right)}.$$

However, $G(k)$ is an increasing function for $k > 1$. Hence, $a = G(1) = \frac{8(N-1)(N^2-2N-2)}{(N^2-4)^2}$ and the proof is completed. $\blacksquare$

Lemma 3.2 *Let $u_0 \in C_0^\infty([0, D])$. Then, the following inequality holds*

$$I_{B_D}[u_0] \geq c \left(\int_{B_D} |u_0|^{\frac{2N}{N-4}} X^{\frac{2N-4}{N-4}} \left(\frac{|x|}{D} \right) dx \right)^{\frac{N-4}{N}}, \tag{3.2}$$

for some positive constant c.

Proof Assume that $D = 1$. From (2.23) we have that

$$\begin{aligned}
I_{B_D}[u_0] &\geq c \int_{B_1} |x|^{4-N} |\Delta u_0|^2 \, dx = c \int_0^1 r^3 \left(u_0'' + \frac{N-1}{r} u_0' \right)^2 dr \\
&= c \left[\int_0^1 r^3 (u_0'')^2 \, dr + (N-1)(N-2) \int_0^1 r(u_0')^2 \, dr \right] \\
&= c \left[\int_{B_1(\mathbb{R}^4)} (\nabla u_0')^2 \, dx + (N-1)(N-2) \int_{B_1(\mathbb{R}^4)} \frac{(u_0')^2}{|x|^2} \, dx \right]. \tag{3.3}
\end{aligned}$$

Applying now the Hardy inequality we have that

$$\int_{B_1(\mathbb{R}^4)} (\nabla u_0')^2 \, dx \geq \left(\frac{4-2}{2} \right)^2 \int_{B_1(\mathbb{R}^4)} \frac{(u_0')^2}{|x|^2} \, dx \geq c \int_0^1 r(u_0')^2 \, dr. \tag{3.4}$$

So, from (3.3) and (3.4) we obtain that

$$I[u_0] \geq c \int_0^1 r(u_0')^2 \, dr. \tag{3.5}$$

Next, we consider the following inequality

$$\int_0^1 r(u_0')^2 \, dr \geq c \left(|u|^q r^{-1} X^{1+q/2}(r) \, dr \right)^{2/q} \tag{3.6}$$

which is implied from [M1, Theorem 3, p. 44] with $X(t) = (-\log t)^{-1}$, $d\nu = r\chi_{[0,1]}dr$ and $d\mu = r^{-1} X^\alpha \chi_{[0,1]}dr$. Setting now $q = \frac{2N}{N-4}$, $\alpha = \frac{2N-4}{N-4}$ and taking into account (3.5) we conclude that

$$I[u] \geq c \left(\int_{B_1} |u_0|^{\frac{2N}{N-4}} X^{\frac{2N-4}{N-4}} \, dx \right)^{\frac{N-4}{N}}. \tag{3.7}$$

Following the same arguments we may prove that (3.7) holds for any B_D, $D > 0$. ∎

Using now Lemma 3.2 we prove inequality (3.2) for every $u \in C_0^\infty(\Omega)$.

Proof of (1.9) From inequality (3.1) we have that

$$I_\Omega[u] \geq I_{B_D}[u_0] + c \int_{B_D} |\Delta(u - u_0)|^2 \, dx. \tag{3.8}$$

The Sobolev imbedding and the fact that X is a bounded function imply that

$$\begin{aligned}
\int_{B_D} |\Delta(u - u_0)|^2 \, dx &\geq c \left(\int_{B_D} |u - u_0|^{\frac{2N}{N-4}} \, dx \right)^{\frac{N-4}{N}} \\
&\geq c \left(\int_{B_D} |u - u_0|^{\frac{2N}{N-4}} X^{\frac{2N-4}{N-4}} \left(\frac{|x|}{D} \right) dx \right)^{\frac{N-4}{N}}. \tag{3.9}
\end{aligned}$$

Then from (3.2), (3.8) and (3.9) we conclude that

$$I_\Omega[u] \geq c \left(\int_{B_D} |u|^{\frac{2N}{N-4}} X^{\frac{2N-4}{N-4}} (\frac{|x|}{D}) \, dx \right)^{\frac{N-4}{N}} = c \left(\int_\Omega |u|^{\frac{2N}{N-4}} X^{\frac{2N-4}{N-4}} (\frac{|x|}{D}) \, dx \right)^{\frac{N-4}{N}}. \quad \blacksquare$$

Theorem 3.3 *Let Ω be a bounded domain, $D = \sup_{x \in \Omega} |x|$ and $u \in C_0^\infty(\Omega)$. Then,*

$$\mathbb{I}_\Omega[u] \geq \mathbb{I}_{B_D}[u_0] + \frac{4(N-1)(N^2 - 4N - 4)}{(N^2 - 4)^2} \int_{B_D} |\Delta(u - u_0)|^2 \, dx. \tag{3.10}$$

Proof Using the fact that $\mathbb{I}_\Omega[u] = \mathbb{I}_{B_D}[u_0] + \sum_{k=1}^\infty \mathbb{I}_{B_D}[u_k]$, it suffices to prove that for any $k = 1, \ldots,$ holds that

$$\mathbb{I}_{B_D}[u_k] \geq \frac{4(N-1)(N^2 - 4N - 4)}{(N^2 - 4)^2} \int_{B_D} |\Delta u_k|^2 \, dx.$$

The result follows from (2.7) and (2.10) using (2.15) and (2.16). $\blacksquare$

Lemma 3.4 *Let $u_0 \in C_0^\infty([0, D])$. Then the following inequality holds*

$$\mathbb{I}_{B_D}[u_0] \geq c \left(\int_{B_D} |\nabla u_0|^{\frac{2N}{N-2}} X^{1 + \frac{N}{N-2}} (\frac{|x|}{D}) \, dx \right)^{\frac{N-2}{N}}, \tag{3.11}$$

for some positive constant c.

Proof Assume that $D = 1$. Making some simple calculations we may obtain that

$$
\begin{aligned}
\mathbb{I}[u_0] &= \int_0^1 r^{N-1} (u_0'' + \frac{N-1}{r} u_0')^2 \, dr - \frac{N^2}{4} \int_0^1 r^{N-3} (u_0')^2 \, dr \\
&= \int_{B_1} (u_0'')^2 \, dx - \left(\frac{N-2}{2} \right)^2 \int_{B_1} (u_0')^2 \, dx \\
&= \int_{B_1} |\nabla w|^2 \, dx - \left(\frac{N-2}{2} \right)^2 \int_{B_1} \frac{w^2}{|x|^2} \, dx,
\end{aligned}
$$

where $w = u_0'$. Using now the following inequality (see [FT, Theorem A])

$$\int_{B_1} |\nabla w|^2 \, dx - \left(\frac{N-2}{2} \right)^2 \int_{B_1} \frac{w^2}{|x|^2} \, dx \geq c \left(\int_\Omega |w|^{\frac{2N}{N-2}} X^{1 + \frac{N}{N-2}} \, dx \right)^{\frac{N-2}{N}}$$

which hold for any $w \in H_0^1(B_1)$, we obtain that (3.11) holds for any $u(r) \in C_0^\infty(B_1)$. Then, following the same arguments we may obtain that (3.11) hold for any B_D, $D > 0$. $\blacksquare$

Proof of (1.10) As in the proof of (1.9) the result is a consequence of Proposition 3.3, Lemma 3.4 and of the following inequality

$$
\begin{aligned}
\int_{B_D} |\Delta(u - u_0)|^2 \, dx &\geq c \left(\int_{B_D} |\nabla u - \nabla u_0|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \\
&\geq c \left(\int_{B_D} |\nabla u - \nabla u_0|^{\frac{2N}{N-2}} X^{1 + \frac{N}{N-2}} (\frac{|x|}{D}) \, dx \right)^{\frac{N-2}{N}},
\end{aligned}
$$

which is implied from the Sobolev imbedding and the fact that X is a bounded function. $\blacksquare$

Proof of Theorem 1.4 Inequality (1.17) is an immediate consequence from Proposition 2.4 and Inequality (1.3). The fact that $\left(1 + \frac{N(N-4)}{8}\right)$ is the best constant will be establish in Section 4. ∎

Proof of Theorem 1.5 We decompose u into spherical harmonics. Then, the result is an immediate consequence of (1.3), (2.7) and (2.14), with $V(|x|) = \frac{1}{4}(N^2 + X_1^2(\frac{|x|}{D}) \ldots X_i^2(\frac{|x|}{D}))$. The fact that $\frac{N^2}{4}$ and $\frac{1}{4}$ are the best constants will be establish in Section 4. ∎

4 Best Constants

Throughout this section we may assume that Ω is a bounded domain, such that $B_1(0) \subset \Omega$ and $N \geq 5$. We initially establish that the constants appearing in the inequalities of Section 2.2 are the best ones. For some $\epsilon > 0$ and $0 < a_1$ we introduce the minimizing sequences u^ϵ and v^ϵ to be defined as:

$$u^\epsilon := r^{-\frac{N-4}{2}+\epsilon} X_1^{\frac{-1+a_1}{2}} \phi(r), \qquad v^\epsilon := r^{\frac{N-4}{2}} u^\epsilon = r^\epsilon X_1^{\frac{-1+a_1}{2}} \phi(r),$$

where $X_1(t) = (1 - \log t)^{-1}$ and $\phi(r) \in C_0^\infty(B_1)$ is a smooth cutoff function, such that $0 \leq \phi \leq 1$, with $\phi \equiv 1$ in $B_{1/2}$.

Lemma 4.1 *As $\epsilon \to 0^+$ and $a_1 \to 0^+$, we have*

$$i) \quad \frac{1}{c_N} \int_\Omega |x|^{2-N} |\nabla v^\epsilon|^2 \, dx = -\frac{-1+a_1}{4} \int_0^1 r^{-1+2\epsilon} X_1^{1+a_1} \phi^2(r) \, dr + O(1), \tag{4.1}$$

$$ii) \quad \frac{1}{c_N} \int_\Omega |x|^{4-N} |\Delta v^\epsilon|^2 \, dx = -\frac{-1+a_1}{4}(N-2)^2 \int_0^1 r^{2\epsilon-1} X_1^{1+a_1} \phi^2 \, dr + O(1), \tag{4.2}$$

$$iii) \quad \frac{1}{c_N} \int_\Omega |x|^{-2} |\nabla u^\epsilon|^2 \, dx = -\frac{-1+a_1}{4} \int_0^1 r^{2\epsilon-1} X_1^{1+a_1} \phi^2 \, dr$$
$$+ \left(\frac{N-4}{2}\right)^2 \int_0^1 r^{2\epsilon-1} X_1^{-1+a_1} \phi^2 \, dr + O(1), \tag{4.3}$$

$$iv) \quad \frac{1}{c_N} \int_\Omega |\Delta u^\epsilon|^2 \, dx = -\frac{-1+a_1}{8}(N^2 - 4N + 8) \int_0^1 r^{2\epsilon-1} X_1^{1+a_1} \phi^2 \, dr$$
$$+ \left(\frac{N(N-4)}{4}\right)^2 \int_0^1 r^{2\epsilon-1} X_1^{-1+a_1} \phi^2 \, dr + O(1), \tag{4.4}$$

$$\tag{4.5}$$

$$v) \quad \frac{1}{c_N} I[u^\epsilon] = -\frac{-1+a_1}{8}(N^2 - 4N + 8) \int_0^1 r^{2\epsilon-1} X_1^{1+a_1} \phi^2 \, dr + O(1), \tag{4.6}$$

$$vi) \quad \frac{1}{c_N} \mathbb{I}[u^\epsilon] = -\frac{-1+a_1}{16}(N-4)^2 \int_0^1 r^{2\epsilon-1} X_1^{1+a_1} \phi^2 \, dr + O(1), \tag{4.7}$$

where c_N is the volume of the unit sphere in $\mathbb{R}^N$.

Proof The conclusion follows from the properties of the functions X_1, ϕ and standard arguments based on integration by parts which also imply that

$$\epsilon \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \phi^2(r) \, dr = -\frac{-1+a_1}{2} \int_0^1 r^{-1+2\epsilon} X_1^{a_1} \phi^2(r) \, dr + O(1)$$

and

$$2\epsilon \int_0^1 r^{-1+2\epsilon} X_1^{a_1} \phi^2(r) \, dr = -a_1 \int_0^1 r^{-1+2\epsilon} X_1^{1+a_1} \phi^2(r) \, dr + O(1).$$

Theorem 4.2 *The quantities*

$$i) \quad 4 + \frac{N(N-4)}{2} \qquad in\ inequality\ (2.17),$$

$$ii) \quad \frac{1}{2} + \frac{2}{(N-2)^2} \qquad in\ inequality\ (2.23),$$

$$iii) \quad \left(\frac{N-4}{2}\right)^2 \qquad in\ inequality\ (2.18),$$

$$iv) \quad 2(N-2)^2 \qquad in\ inequality\ (2.21),$$

$$v) \quad \left(\frac{N-4}{2(N-2)}\right)^2 \qquad in\ inequality\ (2.24),$$

$$vi) \quad \frac{N^2}{4} \qquad in\ inequality\ (1.8),$$

are the best constants.

Proof i) Relations (4.1) and (4.6) imply that

$$\frac{I[u^\epsilon]}{\int_\Omega |x|^{2-N}|\nabla v^\epsilon|^2\,dx} \quad = \quad \frac{\frac{-1+a_1}{8}(N^2-4N+8)\int_0^1 r^{2\epsilon-1}X_1^{1+a_1}\phi^2\,dr + O(1)}{-\frac{-1+a_1}{4}\int_0^1 r^{-1+2\epsilon}X_1^{1+a_1}\phi^2(r)\,dr + O(1)}$$

$$\to \quad 4 + \frac{N(N-4)}{2},$$

as $\epsilon \downarrow 0$ and $a_1 \downarrow 0$. In the same way the conclusion follows for the cases ii) - v). For the last case, observe that

$$\frac{\int_0^1 r^{2\epsilon-1}X_1^{-1+a_1}\phi^2\,dr}{\int_0^1 r^{2\epsilon-1}X_1^{1+a_1}\phi^2\,dr} \to \infty,$$

as $\epsilon \downarrow 0$ and $a \downarrow 0$. Then, from (4.3) and (4.4) we derive that

$$\frac{\int_\Omega |\Delta u^\epsilon|^2\,dx}{\int_\Omega \frac{|\nabla u^\epsilon|^2}{|x|^2}\,dx} \quad = \quad \frac{\frac{-1+a_1}{8}(N^2-4N+8)\int_0^1 r^{2\epsilon-1}X_1^{1+a_1}\phi^2\,dr + \left(\frac{N(N-4)}{4}\right)^2\int_0^1 r^{2\epsilon-1}X_1^{-1+a_1}\phi^2\,dr}{\frac{-1+a_1}{4}\int_0^1 r^{2\epsilon-1}X_1^{1+a_1}\phi^2\,dr + \left(\frac{N-4}{2}\right)^2\int_0^1 r^{2\epsilon-1}X_1^{-1+a_1}\phi^2\,dr}$$

$$\to \quad \frac{N^2}{4},$$

as $\epsilon \downarrow 0$ and $a_1 \downarrow 0$. ∎

Next we complete the proofs of Theorems 1.4 and 1.5. We introduce the minimizing sequences for the k-Improved Hardy-Rellich inequalities. For small positive parameters $\epsilon, a_1, a_2, ..., a_k$ we define

$$u(x) := w(x)\,\phi(|x|), \quad w(x) := |x|^{-\frac{N-4}{2}+\epsilon}X_1^{\frac{-1+a_1}{2}}X_2^{\frac{-1+a_2}{2}}\cdots X_k^{\frac{-1+a_k}{2}},$$

where ϕ is the previous test function and $X_m = X_1(X_{m-1})$, $m = 2, ..., k$. To prove the results we shall estimate the corresponding Rayleigh quotients of u (1.18), (1.20) in the limit $\epsilon \to 0$, $a_1 \to 0$, ..., $a_k \to 0$ in this order.

In the sequel we shall repeatedly use the differentiant rule

$$\frac{d}{dt}X_i^\beta(t) = \frac{\beta}{t}X_1 X_2 \cdots X_{i-1}X_i^{1+\beta}, \quad \beta \neq -1,\ i = 1, 2, ...,$$

and with integrals of the form

$$Q = \int_0^1 r^{-1+2\epsilon} X_1^{1+\beta_1} X_2^{1+\beta_2} \cdots X_k^{1+\beta_k} \phi^2(r) \, dr.$$

For this we notice that

$$Q < \infty \Leftrightarrow \begin{cases} \epsilon > 0, \quad or, \\ \epsilon = 0 \text{ and } \beta_1 > 0, \quad or, \\ \epsilon = 0, \ \beta_1 = 0 \text{ and } \beta_1 > 0, \quad or, \\ \vdots \\ \epsilon = 0, \ \beta_1 = 0, \ ..., \ \beta_{k-1} = 0 \text{ and } \ \beta_k > 0. \end{cases}$$

Also as we pass to the limit $\epsilon \to 0, \ a_1 \to 0, \ ..., \ a_k \to 0$ we have

$$\begin{aligned}
\int_\Omega |\Delta u|^2 \, dx &= \int_\Omega |\Delta w|^2 \phi^2 \, dx + O(1), \\
\int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx &= \int_\Omega \frac{|\nabla w|^2}{|x|^2} \phi^2 \, dx + O(1), \\
\int_\Omega \frac{|\nabla u|^2}{|x|^2} X_1^2 \cdots X_i^2 \, dx &= \int_\Omega \frac{|\nabla w|^2}{|x|^2} X_1^2 \cdots X_i^2 \phi^2 \, dx + O(1), \quad i = 1, ..., k.
\end{aligned}$$

It is not difficult to see that

$$\nabla w(x) = |x|^{-\frac{N-2}{2}+\epsilon} X_1^{\frac{-1+a_1}{2}} X_2^{\frac{-1+a_2}{2}} \cdots X_k^{\frac{-1+a_k}{2}} \left[-\frac{N-4}{2} + \epsilon + \frac{1}{2}\eta(x) \right] \frac{x}{|x|},$$

where $\eta(x) = (-1 + a_1)X_1 + (-1 + a_2)X_1 X_2 + ... + (-1 + a_k)X_1 \cdots X_k$ and

$$\Delta w(x) = \frac{1}{4}|x|^{-\frac{N}{2}+\epsilon} X_1^{\frac{-1+a_1}{2}} X_2^{\frac{-1+a_2}{2}} \cdots X_k^{\frac{-1+a_k}{2}} \left[-N(N-4) + 8\epsilon + 4\epsilon^2 + 4(1+\epsilon)\eta(x) + \eta^2(x) + 2B(x) \right],$$

where

$$\begin{aligned}
B(|x|) &= (-1 + a_1)X_1^2 + (-1 + a_2)(X_1^2 X_2 + X_1^2 X_2^2) + \ldots \\
&\quad + (-1 + a_k)(X_1^2 X_2 \cdots X_k + \ldots + X_1^2 X_2^2 \cdots X_k^2) \\
&= \sum_{i=1}^k (-1 + a_i)X_1^2 \cdots X_i^2 + \sum_{i=2}^k \sum_{j=1}^{i-1} (-1 + a_i)X_1^2 \cdots X_j^2 X_{j+1} \cdots X_i \\
&= \sum_{i=1}^k (-1 + a_i)X_1^2 \cdots X_i^2 + \sum_{j=1}^{k-1} \sum_{i=j+1}^k (-1 + a_i)X_1^2 \cdots X_j^2 X_{j+1} \cdots X_i,
\end{aligned}$$

Note also that

$$r\eta'(r) = B(r)$$

and

$$\eta^2(x) = \sum_{i=1}^k (-1 + a_i)^2 X_1^2 \cdots X_i^2 + 2\sum_{j=1}^{k-1} \sum_{i=j+1}^k (-1 + a_i)(-1 + a_j)X_1^2 \cdots X_j^2 X_{j+1} \cdots X_i.$$

Then we have that

$$
\begin{aligned}
\int_\Omega |\Delta u|^2 dx &= \int_\Omega \tfrac{w^2}{|x|^4}\Bigg[\left(-\tfrac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)^2 + (1+\epsilon)^2\eta^2 + (\tfrac{1}{4}\eta^2 + \tfrac{1}{2}B)^2 \\
&\quad +2(1+\epsilon)\left(-\tfrac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)\eta + 2\left(-\tfrac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)(\tfrac{1}{4}\eta^2 + \tfrac{1}{2}B) \\
&\quad +2(1+\epsilon)(\tfrac{1}{4}\eta^2 + \tfrac{1}{2}B)\eta\Bigg]\phi^2\,dx \\
&= \int_\Omega \tfrac{w^2}{|x|^4}\Bigg[\left(-\tfrac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)^2 + (1+\epsilon)^2\eta^2 \\
&\quad +2(1+\epsilon)\left(-\tfrac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)\eta + 2\left(-\tfrac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)(\tfrac{1}{4}\eta^2 + \tfrac{1}{2}B)\Bigg]\cdot \\
&\quad \cdot \phi^2\,dx + O(1),
\end{aligned}
$$

$$
\int_\Omega \tfrac{|\nabla u|^2}{|x|^2}X_1^2\cdots X_i^2\,dx = \int_\Omega \tfrac{w^2}{|x|^4}\left[\left(-\tfrac{N-4}{2} + \epsilon\right)^2 + \left(-\tfrac{N-4}{2} + \epsilon\right)\eta + \tfrac{1}{4}\eta^2\right]X_1^2\cdots X_i^2\,\phi^2\,dx + O(1),
$$

$$
\int_\Omega \tfrac{u^2}{|x|^4}X_1^2\cdots X_i^2\,dx = \int_\Omega \tfrac{w^2}{|x|^4}X_1^2\cdots X_i^2\,\phi^2 dx + O(1).
$$

An important quantity that appears is

$$
\sum_{i=1}^{k} a_i A_i - \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(1-a_j)\Gamma_{ij},
$$

where

$$
\begin{aligned}
A_i(a_1,\ldots a_k) &:= \int_0^1 r^{-1}X_1^{1+a_1}\cdots X_i^{1+a_i}X_{i+1}^{-1+a_{i+1}}\cdots X_k^{-1+a_k}\phi^2\,dr, \quad i=1,\ldots,k \\
\Gamma_{ij}(a_1,\ldots a_k) &:= \int_0^1 r^{-1}X_1^{1+a_1}\cdots X_i^{1+a_i}X_{i+1}^{a_{i+1}}\cdots X_j^{a_j}X_{j+1}^{-1+a_{j+1}}\cdots X_k^{-1+a_k}\phi^2\,dr, \quad i<j.
\end{aligned}
$$

We will pass to the limit initially $a_1 \to 0^+$ and then $a_2 \to 0^+$, ..., $a_{k-1} \to 0^+$. In passing to the limit we will use identities similar to the ones used in Step 8 of [BFT], in particular we have

$$
a_1 A_1 = \int_0^1 (X_1^{a_1})' X_2^{-1+a_2}\cdots X_k^{-1+a_k}\phi^2\,dr = -\sum_{j=2}^{k}(-1+a_j)\Gamma_{ij} + O(1), \quad \text{as } a_1 \downarrow 0.
$$

Therefore,

$$
\sum_{i=1}^{k} a_i A_i - \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(1-a_j)\Gamma_{ij} = \sum_{i=2}^{k} a_i A_i - \sum_{i=2}^{k-1}\sum_{j=i+1}^{k}(1-a_j)\Gamma_{ij} + O(1), \quad \text{as } a_1 \downarrow 0 \tag{4.8}
$$

Then we pass to the limit $a_1 \to 0$, in the right hand side of (4.8). Again we use the identity

$$
\begin{aligned}
a_2 A_2(0, a_2, ..., a_k) &= \int_0^1 (X_2^{a_2})' X_3^{-1+a_3}\cdots X_k^{-1+a_k}\phi^2\,dr \\
&= -\sum_{j=3}^{k}(-1+a_j)\Gamma_{ij} + O(1), \quad \text{as } a_2 \downarrow 0.
\end{aligned}
$$

and therefore by iterating the previous procedure we pass to the limit $a_2 \to 0^+$, ..., $a_{k-1} \to 0^+$ to conclude

$$\sum_{i=1}^{k} a_i A_i - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (1 - a_j) \Gamma_{ij} \;=\; a_k A_k(0, 0, \ldots, 0, a_k) + O(1),$$

$$= a_k \int_0^1 r^{-1} X_1 X_2 \cdots X_{k-1} X_k^{-1+a_k} \phi^2 \, dr + O(1), \text{ as } a_k \downarrow 0. \quad (4.9)$$

Completion of Proof of Theorem 1.4. We use the previous test functions to conclude that

$$R[u] := \int_\Omega |\Delta u|^2 dx - \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4} dx - \left(1 + \frac{N(N-4)}{8}\right) \sum_{i=1}^{k-1} \int_\Omega \frac{u^2}{|x|^4} X_1^2 \cdots X_i^2 \, dx =$$

$$= \int_\Omega \frac{w^2}{|x|^4} \left[\epsilon^2 (2+\epsilon)^2 - \frac{N(N-4)}{2}\epsilon(2+\epsilon) + 2(1+\epsilon)\left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)\eta \right.$$

$$+ \left(1 - \frac{N(N-4)}{8} + 3\epsilon + \frac{3}{2}\epsilon^2\right)\eta^2 + \left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right) B$$

$$\left. - \left(1 + \frac{N(N-4)}{8}\right) \sum_{i=1}^{k-1} X_1^2 \cdots X_i^2 \right] \phi^2 \, dx + O(1),$$

$$= c_N \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \left[\epsilon^2 (2+\epsilon)^2 - \frac{N(N-4)}{2}\epsilon(2+\epsilon) \right.$$

$$+ 2(1+\epsilon)\left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)\eta + \left(1 - \frac{N(N-4)}{8} + 3\epsilon + \frac{3}{2}\epsilon^2\right)\eta^2$$

$$\left. + \left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right) B - \left(1 + \frac{N(N-4)}{8}\right) \sum_{i=1}^{k-1} X_1^2 \cdots X_i^2 \right] \phi^2 \, dr + O(1),$$

But

$$2\epsilon \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \phi^2(r) dr \;=\; -\int_0^1 r^{2\epsilon} (X_1^{-1+a_1} \cdots X_k^{-1+a_k})' \phi^2(r) dr + O(1)$$

$$= -\int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \eta \phi^2(r) dr + O(1). \quad (4.10)$$

and

$$2\epsilon \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \eta \phi^2(r) dr = -\int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \eta^2 \phi^2(r) dr$$

$$- \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} B \phi^2(r) dr + O(1). \quad (4.11)$$

Therefore

$$R[u] = c_N \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \left[\epsilon^3 + \epsilon^4 + (6 + 2\epsilon)\epsilon^2 \eta + \left(3\epsilon + \frac{3}{2}\epsilon^2\right)\eta^2 \right.$$

$$\left. + \left(-1 - \frac{N(N-4)}{8} + 2\epsilon + \epsilon^2\right) B - \left(1 + \frac{N(N-4)}{8}\right) \sum_{i=1}^{k-1} X_1^2 \cdots X_i^2 \right] \phi^2 \, dr + O(1),$$

passing to the limit $\epsilon \to 0$, we obtain

$$\frac{1}{c_N} R[u] \;=\; \left(1 + \frac{N(N-4)}{8}\right) \int_0^1 r^{-1} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \left[B + \sum_{i=1}^{k-1} X_1^2 \cdots X_i^2\right] \phi^2 \, dr + O(1)$$

$$= \left(1 + \frac{N(N-4)}{8}\right) \int_0^1 r^{-1} X_1^{1+a_1} \cdots X_k^{1+a_k} \phi^2 \, dr -$$

$$- \left(1 + \frac{N(N-4)}{8}\right) \int_0^1 r^{-1} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \left[\sum_{i=1}^{k} a_i X_1^2 \cdots X_i^2 + \right.$$

$$\left. + \sum_{j=1}^{k-1} \sum_{i=j+11}^{k} (-1 + a_i) X_1^2 \cdots X_j^2 X_{j+1} \cdots X_i \right] \phi^2 \, dr + O(1).$$

or

$$\frac{1}{c_N}R[u] = \left(1 + \frac{N(N-4)}{8}\right)A_k - \left(1 + \frac{N(N-4)}{8}\right)\left(\sum_{i=1}^{k} a_i A_i - \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(1 - a_j)\Gamma_{ij}\right) + O(1).$$

However, we can pass to the limit $a_1 \downarrow 0, \ldots a_{k-1} \downarrow 0$ see (4.9), to conclude that

$$\frac{1}{c_N}R[u] = \left(1 + \frac{N(N-4)}{8}\right)A_k - \left(1 + \frac{N(N-4)}{8}\right)a_k A_k + O(1), \text{ as } a_k \downarrow 0.$$

The Rayleigh quotient now of (1.18) is smaller or equal than

$$\frac{\left(1 + \frac{N(N-4)}{8}\right)A_k - \left(1 + \frac{N(N-4)}{8}\right)a_k A_k + O(1)}{A_k} \to 1 + \frac{N(N-4)}{8},$$

since $A_k \to \infty$, as $a_k \downarrow 0$. ∎

Completion of Proof of Theorem 1.5. Once more we use the same minimizing sequence to conclude

$$\int_\Omega |\Delta u|^2 dx - \frac{N^2}{4}\int_\Omega \frac{|\nabla u|^2}{|x|^2}dx - \frac{1}{4}\sum_{i=1}^{k-1}\int_\Omega \frac{|\nabla u|^2}{|x|^2}X_1^2 \cdots X_i^2\, dx =$$

$$= c_N \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_i^{-1+a_k}\left[\left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)^2 + (1+\epsilon)^2\eta^2\right.$$

$$+2(1+\epsilon)\left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)\eta + 2\left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)(\tfrac{1}{4}\eta^2 + \tfrac{1}{2}B)$$

$$-\frac{N^2}{4}\left(-\frac{N-4}{2} + \epsilon\right)^2 - \frac{N^2}{4}\left(-\frac{N-4}{2} + \epsilon\right)\eta - \frac{N^2}{16}\eta^2$$

$$\left.-\frac{1}{4}\left(-\frac{N-4}{2} + \epsilon\right)^2\sum_{i=1}^{k-1}X_1^2 \cdots X_i^2\right]\phi^2\, dr + O(1),$$

$$= c_N \int_0^1 r^{-1+2\epsilon} X_1^{-1+2a_1} \cdots X_i^{-1+2a_k}\left[\frac{N(N-4)^2}{4}\epsilon + \left(4 - \frac{N(N-4)}{2} - \frac{N^2}{4}\right)\epsilon^2 + 4\epsilon^3 + \epsilon^4\right.$$

$$+\left(\frac{N(N-4)^2}{8} + \left(4 - \frac{N(N-4)}{2} - \frac{N^2}{4}\right)\epsilon + 6\epsilon^2 + 2\epsilon^3\right)\eta$$

$$+\left(1 - \frac{N(N-4)}{8} - \frac{N^2}{16} + 3\epsilon + \tfrac{3}{2}\epsilon^2\right)\eta^2 + \left(-\frac{N(N-4)}{4} + 2\epsilon + \epsilon^2\right)B$$

$$\left.-\frac{1}{4}\left(-\frac{N-4}{2} + \epsilon\right)^2\sum_{i=1}^{k-1}X_1^2 \cdots X_i^2\right]\phi^2\, dr + O(1),$$

We now use identities (4.10), (4.11) and passing to the limit $\epsilon \to 0$, to conclude that

$$\int_\Omega |\Delta u|^2 dx - \frac{N^2}{4}\int_\Omega \frac{|\nabla u|^2}{|x|^2}dx - \frac{1}{4}\sum_{i=1}^{k-1}\int_\Omega \frac{|\nabla u|^2}{|x|^2}X_1^2 \cdots X_i^2\, dx =$$

$$= -\frac{1}{4}\left(\frac{N-4}{2}\right)^2 c_N \int_0^1 r^{-1} X_1^{-1+a_1} \cdots X_i^{-1+a_k}\left[B + \sum_{i=1}^{k-1}X_1^2 \cdots X_i^2\right]\phi^2\, dr + O(1),$$

$$= \frac{1}{4}\left(\frac{N-4}{2}\right)^2 c_N A_k - \frac{1}{4}\left(\frac{N-4}{2}\right)^2 c_N\left(\sum_{i=1}^{k} a_i A_i - \sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(1 - a_j)\Gamma_{ij}\right) + O(1).$$

As before, using (4.9) we can pass to the limit $a_1 \downarrow 0, ...a_{k-1} \downarrow 0$ see (4.9), the Rayleigh quotient now of (1.20) is smaller or equal than

$$\frac{\frac{1}{4}\left(\frac{N-4}{2}\right)^2 A_k - \frac{1}{4}\left(\frac{N-4}{2}\right)^2 a_k A_k + O(1)}{\left(\frac{N-4}{2}\right)^2 A_k + O(1)} \to \frac{1}{4},$$

since $A_k \to \infty$, as $a_k \downarrow 0$. ∎

5 Existence of minimizers in $W_0^{2,2}(\Omega, |x|^{-(N-4)})$

In this section we assume certain improved inequalities (1.6), (1.8) in v-terms and we prove the existence of minimizers, in some appropriate weighted spaces. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain containing the origin and $N \geq 5$. We introduce the space $W_0^{2,2}(\Omega, |x|^{-(N-4)})$ to be defined as the closure of the C_0^∞ functions with respect to the norm

$$\|v\|_W^2 \quad := \quad \int_\Omega |x|^{-(N-4)}|\Delta v|^2 \, dx + \int_\Omega |x|^{-N}|x \cdot \nabla v|^2 \, dx + \int_\Omega |x|^{-(N-2)}|\nabla v|^2 \, dx$$

$$+ \int_\Omega |x|^{-N} v^2 \, dx. \tag{5.1}$$

It is clear that $W_0^{2,2}(\Omega, |x|^{-(N-4)})$ is a Hilbert space with inner product

$$< \phi, \psi >_W \quad := \quad \int_\Omega |x|^{-(N-4)}\Delta\phi\,\Delta\psi \, dx + \int_\Omega |x|^{-N}(x \cdot \nabla\phi)(x \cdot \nabla\psi) \, dx$$

$$+ \int_\Omega |x|^{-(N-2)}\nabla\phi \cdot \nabla\psi \, dx + \int_\Omega |x|^{-N}\phi\,\psi \, dx.$$

Lemma 5.1 *i) If $u \in H_0^2(\Omega)$, then $v = |x|^{\frac{N-4}{2}} u \in W_0^{2,2}(\Omega, |x|^{-(N-4)})$.*
  ii) If $v \in C_0^\infty(\Omega)$, then $u = |x|^{-\alpha}v \in H_0^2(\Omega)$, for $\alpha < \frac{N-4}{2}$.

Proof i) Let $u \in H_0^2(\Omega)$. Hardy's inequality (1.6) implies that

$$\int_\Omega |x|^{-N}|v|^2 \, dx = \int_\Omega \frac{|u|^2}{|x|^4} \, dx < \infty.$$

In this direction, from relations (2.17), (2.23) we deduce that

$$\int_\Omega |x|^{-(N-4)}|\Delta v|^2 \, dx \leq c_1 I[u] < \infty,$$

$$\int_\Omega |x|^{-(N-2)}|\nabla v|^2 \, dx \leq c_2 I[u] < \infty,$$

for some positive constants c_1, c_2. Hence $\|v\|_W < \infty$.
  ii) Let $v \in C_0^\infty(\Omega)$ and $u = |x|^{-\alpha}v$, for $\alpha < \frac{N-4}{2}$. It is known (see [FT]) that

$$\left(b - \frac{N-2}{2}\right)^2 \int_\Omega |x|^{-2b-2}w^2 \, dx \leq \int_\Omega |x|^{-2b}|\nabla w|^2 \tag{5.2}$$

for any $w \in C_0^\infty$ and $b \leq \frac{N-2}{2}$. Inequality (5.2) for $b = a + 1$, $a < \frac{N-4}{2}$, implies that

$$\left(a - \frac{N-4}{2}\right)^2 \int_\Omega |x|^{-2a-4} w^2 \, dx \leq \int_\Omega |x|^{-2a-2} |\nabla w|^2. \tag{5.3}$$

Hence, from (2.3) and (5.3) we conclude that

$$
\begin{aligned}
\int_\Omega |\Delta u|^2 \, dx &\leq C_a \left[\int_\Omega |x|^{-2a} |\Delta v|^2 \, dx + \int_\Omega |x|^{-2a-4} |x \cdot \nabla v|^2 \, dx + \int_\Omega |x|^{-2a-2} |\nabla v|^2 \, dx \right] \\
&\leq C_a \left[\int_\Omega |x|^{-(N-4)} |\Delta v|^2 \, dx + \int_\Omega |x|^{-N} |x \cdot \nabla v|^2 \, dx + \int_\Omega |x|^{-(N+2)} |\nabla v|^2 \, dx \right] \\
&\leq C_a \, \|v\|_{W_0^{2,2}(\Omega, |x|^{-(N-4)})} < \infty
\end{aligned}
$$

and the proof is completed. ∎

Lemma 5.2 *The functionals J, $\mathbb{J}$ are weakly lower semicontinuous in $W_0^{2,2}(\Omega, |x|^{-(N-4)})$.*

Proof Let v_n be a weakly convergent sequence in $W_0^{2,2}(\Omega, |x|^{-(N-4)})$, to some v_0. Assume also the sequence $w_n := v_n - v_0$, with $w_n \rightharpoonup 0$. Then for n large enough, we may prove that

$$
\begin{aligned}
J(v_n) &= J(w_n + v_0) = \\
&= \int_\Omega |x|^{4-N} |\Delta w_n + \Delta v_0|^2 \, dx - N(N-4) \int_\Omega |x|^{-N} |x \nabla w_n + x \nabla v_0|^2 \, dx \\
&\quad + \frac{N(N-4)}{2} \int_\Omega |x|^{2-N} |\nabla w_n + \nabla v_0|^2 \, dx = \\
&= J(w_n) + J(v_0) + o(1).
\end{aligned}
$$

Since $J(w_n) \geq 0$, we conclude that $\liminf_{n\to\infty} J(v_n) \geq J(v_0)$. The case of $\mathbb{J}$ may be treated in a similar way. ∎

5.1   Existence of minimizers for improved inequalities of (1.6)

Assume the improved Hardy inequality

$$\int_\Omega |\Delta u|^2 \, dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4} \, dx + b \int_\Omega V \, u^2 \, dx. \tag{5.4}$$

We want the potential V to be a lower order potential compared to the Hardy potential $\frac{1}{|x|^4}$. For that reason we give the following definition of the admissible class $\mathcal{A}$ of potentials : We say that a potential V is an admissible potential, that is $V \in \mathcal{A}$, if V is not everywhere nonpositive, $V \in L_{loc}^{\frac{N}{4}}(\Omega \setminus \{0\})$, and there exists a positive constant c, such that

$$\int_\Omega |\Delta u|^2 dx \geq \left(\frac{N(N-4)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^4} dx + c \int_\Omega |V| u^2 dx, \quad \text{for any } u \in H_0^2(\Omega). \tag{5.5}$$

The presence of the absolute value in the right hand side of (5.5) ensures that the negative part of V is itself a lower order potential compared to the Hardy potential, and therefore the Hardy potential is truly present in (1.6). As a consequence of (1.9), the class $\mathcal{A}$ contains all non everywhere nonpositive potentials V, such that $\int_\Omega |V|^{\frac{N}{4}} X^{1-N/2} dx < \infty$.

Actually, the best constants arising in the inequalities of type (5.4) in u-terms are equal with those ones arising in the corresponding inequalities in v-terms ($v = |x|^{\frac{N-4}{2}} u$). For example, we have:

Lemma 5.3 *The best constants*

$$c := \inf_{\substack{u \in H_0^2(\Omega), \\ \int_\Omega V u^2 \, dx > 0}} \frac{I[u]}{\int_\Omega |V| u^2 \, dx} \tag{5.6}$$

and

$$C := \inf_{\substack{v \in W_0^{2,2}(\Omega, |x|^{-(N-4)}), \\ \int_\Omega |x|^{-(N-4)} V v^2 \, dx > 0}} \frac{J(v)}{\int_\Omega |x|^{-(N-4)} |V| v^2 \, dx} \tag{5.7}$$

are equal.

Proof Let c, C be the best constants in (5.6) and (5.7), respectively. For any $u \in C_0^\infty(\Omega)$ and $v = |x|^{\frac{N-4}{2}} u$, Lemma 2.3 implies that

$$\frac{I[u]}{\int_\Omega |V| u^2 \, dx} = \frac{J(v)}{\int_\Omega |x|^{-(N-4)} |V| v^2 \, dx}.$$

Hence, $c \geq C$. Next we claim that $c \leq C$. Fix $\epsilon > 0$ and assume the functions $v_\epsilon \in C_0^\infty(\Omega)$, such that

$$\frac{J(v_\epsilon)}{\int_\Omega |x|^{-(N-4)} |V| v_\epsilon^2 \, dx} \leq C + \epsilon.$$

Let $0 < a < \frac{N-4}{2}$. Lemma 5.1 implies that $u_{a,\epsilon} = |x|^{-a} v_\epsilon \in H_0^2(\Omega)$ providing that

$$c \leq \frac{I[u_{a,\epsilon}]}{\int_\Omega |V| u_{a,\epsilon}^2 \, dx} = \frac{J_a(v_\epsilon)}{\int_\Omega |x|^{-2a-4} |V| v_\epsilon^2 \, dx}, \tag{5.8}$$

where

$$\begin{aligned}
J_a(v) \quad &:= \quad \int_\Omega |x|^{-2a} |\Delta v|^2 \, dx - 4a(a+2) \int_\Omega |x|^{-2a-4} |x \cdot \nabla v|^2 \, dx \\
&\quad + 2a(a+2) \int_\Omega |x|^{-2a-2} |\nabla v|^2 \, dx \\
&\quad + \left[a(a+2)(-N+a+2)(-N+a+4) - \left(\frac{N(N-4)}{4} \right)^2 \right] \int_\Omega |x|^{-2a-4} v^2 \, dx.
\end{aligned}$$

Next we calculate the limit of $J_a(v)$ as $a \to \frac{N-4}{2}^-$. It is clear that

$$\int_\Omega |x|^{-2a} |\Delta v|^2 \, dx \quad \to \quad \int_\Omega |x|^{-(N-4)} |\Delta v|^2 \, dx,$$

$$\int_\Omega |x|^{-2a-2} |\nabla v|^2 \, dx \quad \to \quad \int_\Omega |x|^{-(N-2)} |\nabla v|^2 \, dx.$$

as $a \to \frac{N-4}{2}^-$. However, the problem arises in the case of $\lim_{a \to \frac{N-4}{2}^-} \int_\Omega |x|^{-2a-4} v^2 \, dx$. In this case, we have that

$$\left[a(a+2)(-N+a+2)(-N+a+4) - \left(\frac{N(N-4)}{4} \right)^2 \right] \cdot \int_\Omega |x|^{-2a-4} v^2 \, dx$$

$$\leq C \, ||v||_\infty \frac{a(a+2)(-N+a+2)(-N+a+4) - \left(\frac{N(N-4)}{4} \right)^2}{N - 2a - 4} \to 0,$$

as $a \to \frac{N-4}{2}^-$, hence

$$\lim_{a \to \frac{N-4}{2}^-} J_a(v) = J(v),$$

for any $v \in C_0^\infty(\Omega/\{0\})$. Taking the limit $a \to \frac{N-4}{2}^-$ in (5.8), we obtain that

$$c \leq C + \epsilon,$$

for any fixed $\epsilon > 0$ and the proof is completed. ∎

By the same argument the Hardy-Sobolev inequality (1.9) takes the following form:

Lemma 5.4 *Let $D \geq \sup_{x \in \Omega} |x|$. Then, there exists $c > 0$, such that*

$$J[v] \geq c \left(\int_\Omega |x|^{-N} |v|^{\frac{2N}{N-4}} X^{\frac{2N-4}{N-2}} (\frac{|x|}{D}) dx \right)^{\frac{N-4}{N}}, \tag{5.9}$$

for every $v \in W_0^{2,2}(\Omega, |x|^{-(N-4)})$.

Define the following quantity

$$Q[v] := \frac{J(v)}{\int_\Omega |x|^{-(N-4)} V \, v^2 \, dx}$$

and set

$$B := \inf_{\substack{v \in C_0^\infty(\Omega), \\ \int_\Omega |x|^{-(N-4)} V \, v^2 \, dx > 0}} Q[v] = \inf_{\substack{v \in W_0^{2,2}(\Omega, |x|^{-(N-4)}), \\ \int_\Omega |x|^{-(N-4)} V \, v^2 \, dx > 0}} Q[v] \tag{5.10}$$

By practically the same arguments as in Lemma 5.3 we have that

Lemma 5.5 *There holds: $B = b$*

The local best constant of inequality (5.4) can be written as:

$$C^0 := \lim_{r \downarrow 0} C_r, \qquad C_r := \inf_{\substack{v \in C_0^\infty(B_r), \\ \int_{B_r} |x|^{-(N-4)} V \, v^2 \, dx > 0}} \frac{J(v)}{\int_{B_r} |x|^{-(N-4)} V \, v^2 \, dx}. \tag{5.11}$$

If there is no $v \in C_0^\infty(B_r)$, such that $\int_{B_r} |x|^{-(N-4)} V \, v^2 \, dx > 0$, for some $r > 0$, we set $C_r = \infty$. Observe that $B \leq C^0$.

Theorem 5.7 *Let*

$$B < C^0. \tag{5.12}$$

Then B is achieved by some $v_0 \in W_0^{2,2}(\Omega, |x|^{-(N-4)})$.

Proof Let $\{v_k\} \subset W_0^{2,2}(\Omega, |x|^{-(N-4)})$ be a minimizing sequence for (5.10), such that

$$\int_\Omega |x|^{-(N-4)} V \, v_k^2 \, dx = 1, \tag{5.13}$$

for every k. Hence $J(v_k) \to B$. Since $J(v_k)$ is bounded, from (2.17) and (2.23) we deduce that $\{v_k\}$ must be bounded too, in $W_0^{2,2}(\Omega, |x|^{-(N-4)})$. Therefore, there exists a subsequence, still denoted by $\{v_k\}$, such that

$$v_k \rightharpoonup v_0, \quad \text{in } W_0^{2,2}(\Omega, |x|^{-(N-4)})$$

and
$$v_k \to v_0, \quad \text{in } L^2(\Omega/B_\rho), \quad \text{for every } \rho > 0,$$

for some $v_0 \in W_0^{2,2}(\Omega, |x|^{-(N-4)})$. We set $w_k := v_k - v_0$. Then from (5.12) we have that

$$1 = \int_\Omega |x|^{-(N-4)} V w_k^2 \, dx + \int_\Omega |x|^{-(N-4)} V v_0^2 \, dx + o(1). \tag{5.14}$$

In addition from Lemma 5.2 we deduce that

$$B = J(w_k) + J(v_0) + o(1)$$

or

$$B \geq J(w_k) + B \int_\Omega |x|^{-(N-4)} V v_0^2 \, dx + o(1) \tag{5.15}$$

and

$$B \geq J(v_0). \tag{5.16}$$

Observe also that (5.12) implies the existence of a $\rho > 0$, sufficiently small, such that

$$B \leq C_\rho = \inf_{\substack{v \in C_0^\infty(B_\rho), \\ \int_{B_\rho} |x|^{-(N-4)} V v^2 \, dx > 0}} \frac{J(v)}{\int_{B_\rho} |x|^{-(N-4)} V v^2 \, dx}. \tag{5.17}$$

Assume the cutoff function $\phi \in C_0^\infty(B_\rho)$, such that $0 \leq \phi \leq 1$, in B_ρ and $\phi \equiv 1$, in $B_{\rho/2}$. Set $w_k = \phi w_k + (1 - \phi)w_k$. Making some calculations we have that

$$\begin{aligned}
J(w_k) &= J(\phi w_k) + J((1 - \phi)w_k) + 2 \int_{B_\rho} |x|^{-(N-4)} \Delta(\phi w_k) \Delta((1 - \phi)w_k) \, dx \\
&\quad - N(N-4) \Bigg[2 \int_{B_\rho} |x|^{-N} (x \cdot \nabla(\phi w_k)) (x \cdot \nabla((1 - \phi)w_k)) \, dx \\
&\qquad - \frac{1}{2} \int_{B_\rho} |x|^{-N} \nabla(\phi w_k) \cdot \nabla((1 - \phi)w_k) \, dx \Bigg] \\
&= J(\phi w_k) + J((1 - \phi)w_k) + 2 \int_{B_\rho} |x|^{-(N-4)} \phi(1 - \phi) |\Delta w_k|^2 \, dx + o(1).
\end{aligned}$$

Since $J((1 - \phi)w_k) \geq 0$ we obtain that

$$J(w_k) \geq J(\phi w_k) + o(1). \tag{5.18}$$

From (5.17) we have that

$$J(\phi w_k) \geq C_\rho \int_{B_\rho} |x|^{-(N-4)} V(\phi w_k)^2 \, dx. \tag{5.19}$$

Since $V \in L_{loc}^{N/4}(\Omega/\{0\})$ holds that

$$\int_{\Omega/B_{\rho/2}} |x|^{-(N-4)} V w_k^2 \, dx \to 0, \quad \text{as } k \to \infty. \tag{5.20}$$

So, inequalities (5.18), (5.19) and (5.20) imply that

$$J(w_k) \geq C_\rho \int_\Omega |x|^{-(N-4)} V w_k^2 \, dx + o(1). \tag{5.21}$$

Then, from (5.14) and (5.21) we derive that

$$J(w_k) \geq C_\rho \left(1 - \int_\Omega |x|^{-(N-4)} V v_0^2 \, dx \right) + o(1).$$

Taking into account (5.15) we conclude that

$$B \geq C_\rho \left(1 - \int_\Omega |x|^{-(N-4)} V v_0^2 \, dx \right) + B \int_\Omega |x|^{-(N-4)} V \, v_0^2 \, dx + o(1),$$

or

$$(B - C_\rho) \left(1 - \int_\Omega |x|^{-(N-4)} V v_0^2 \, dx \right) \geq 0$$

which implies that

$$\int_\Omega |x|^{-(N-4)} V v_0^2 \, dx \geq 1$$

and from (5.16) that

$$0 \leq \frac{J(v_0)}{\int_\Omega |x|^{-(N-4)} V v_0^2 \, dx} \leq B.$$

It follows that B is attained by v_0. We note that

$$\int_\Omega |x|^{-(N-4)} V v_0^2 \, dx = 1$$

and it follows from (5.15) that v_k converges strongly in $W_0^{2,2}(\Omega, |x|^{-(N-4)})$ to v_0. ∎

We next look for an improvement of inequality (5.4). That is, for an inequality of the form:

$$\int_\Omega |\Delta u|^2 dx \geq \left(\frac{N(N-4)}{4} \right)^2 \int_\Omega \frac{u^2}{|x|^4} dx + b \int_\Omega V u^2 dx + b_1 \int_\Omega W u^2 dx, \qquad u \in H_0^2(\Omega), \tag{5.22}$$

where V and W are both in $\mathcal{A}$. Assuming that (5.22) holds true, the best constant b_1, is clearly given by:

$$b_1 = \inf_{\substack{u \in H_0^2(\Omega) \\ \int_\Omega W u^2 dx > 0}} \frac{I[u] - b \int_\Omega V u^2 dx}{\int_\Omega W u^2 dx}. \tag{5.23}$$

By the same argument as in Lemma 5.3, the constant b_1 is also equal to:

$$B_1 = \inf_{\substack{v \in W_0^{2,2}(\Omega, |x|^{-(N-4)}) \\ \int_\Omega |x|^{-(N-4)} W v^2 dx > 0}} \frac{J[v] - b \int_\Omega |x|^{-(N-4)} V v^2 dx}{\int_\Omega |x|^{-(N-4)} W v^2 dx}. \tag{5.24}$$

Notice that by the properties of $b = B$ we always have that $b_1 \geq 0$. Conversely, if one defines $b_1 \geq 0$ by (5.24) it is immediate that inequality (5.22) holds true with b_1 being the best constant. But of course, for (5.22) to be an improvement of the original inequality, we need b_1 to be strictly positive. Our next result is a direct consequence of Proposition 5.7 and provides conditions under which the original inequality cannot be improved.

Theorem 5.8 *Suppose that $b < C^0$. Let V and W be both in $\mathcal{A}$. If ϕ is the minimizer of the quotient (5.10) and*

$$\int_\Omega |x|^{-(N-4)} W \phi^2 dx > 0,$$

then $b_1 = 0$, that is, there is no further improvement of (5.4).

Proof By our assumptions, $v = \phi$ is an admissible function in (5.24). Moreover, for $v = \phi$ the numerator of (5.24) becomes zero. In view of the fact that $b_1 \geq 0$, we conclude that $b_1 = 0$. ■

It follows in particular that if $W \geq 0$, we cannot improve (1.6). Thus, the following result has been proved.

Theorem 5.9 *Let $V \in \mathcal{A}$. If*

$$b < C^0,$$

then, we cannot improve (5.4) by adding a nonnegative potential $W \in \mathcal{A}$.

As a consequence of (1.9) and Theorem 5.9 we have:

Corollary 5.10 *Let $D > \sup_{x \in \Omega} |x|$. Suppose V is not everywhere nonpositive and such that*

$$\int_\Omega |V|^{\frac{N}{4}} X^{1-N/2}(|x|/D) \, dx < \infty.$$

Then, $V \in \mathcal{A}$ but there is no further improvement of (5.4) with a nonnegative $W \in \mathcal{A}$.

Proof Applying Holder's inequality we get:

$$\int_\Omega |x|^{-(N-4)} |V| v^2 \, dx \leq \left(\int_\Omega |V|^{\frac{N}{4}} X^{1-N/2} \, dx \right)^{\frac{4}{N}} \left(\int_\Omega |x|^{-N} X^{\frac{2N-4}{N-4}} |v|^{\frac{2N}{N-4}} \, dx \right)^{\frac{N-4}{N}}.$$

The first integral is bounded by our assumption, whereas the second integral is bounded from above by $C \, J[v]$ (cf Lemma 5.4). Thus we proved that $V \in \mathcal{A}$. Using once more Holder's inequality in B_r and the definition of C_r (cf (5.11)) we easily see that:

$$C_r \geq \frac{C}{\left(\int_{B_r} |V|^{\frac{N}{4}} X^{1-N/2} \, dx \right)^{\frac{4}{N}}} \to \infty, \quad \text{as} \quad r \to 0,$$

whence $\mathcal{C}^0 = +\infty$. Thus, all conditions of Theorem 5.9 are satisfied and the result follows. ■

5.2 Existence of minimizers for improved inequalities of (1.8)

Assume the following improved inequality

$$\int_\Omega |\Delta u|^2 \, dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} \, dx + \mathsf{b} \int_\Omega V \, |\nabla u|^2 \, dx. \tag{5.25}$$

We want the potential V to be a lower order potential compared to the Hardy potential $\frac{1}{|x|^2}$. For that reason we give the following definition of the admissible class $\mathbb{A}$ of potentials :

Definition 5.11 *We say that a potential V is an admissible potential, that is $V \in \mathbb{A}$, if V is not everywhere nonpositive, $V \in L^{\frac{N}{2}}_{loc}(\Omega \setminus \{0\})$, and there exists a positive constant c, such that*

$$\int_\Omega |\Delta u|^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^2} dx + c \int_\Omega |V||\nabla u|^2 dx, \quad \text{for any } u \in H_0^2(\Omega). \tag{5.26}$$

The presence of the absolute value in the right hand side of (5.5) ensures that the negative part of V is itself a lower order potential compared to the Hardy potential, and therefore the Hardy potential is truly present in (1.8). As a consequence of (1.10), the class $\mathbb{A}$ contains all non everywhere nonpositive potentials V, such that $\int_\Omega |V|^{\frac{N}{2}} X^{1-N} dx < \infty$.

Lemma 5.12 *The best constants*

$$\mathsf{c} := \inf_{\substack{u \in H_0^2(\Omega), \\ \int_\Omega V\,|\nabla u|^2\,dx > 0}} \frac{\mathbb{I}[u]}{\int_\Omega |V|\,|\nabla u|^2\,dx}, \tag{5.27}$$

and

$$\mathbb{C} := \inf_{\substack{v \in W_0^{2,2}(\Omega, |x|^{-(N-4)}), \\ \int_\Omega |x|^{-(N-4)}V\,|\nabla v - \frac{N-4}{2}\frac{x}{|x|^2}v|^2\,dx > 0}} \frac{\mathbb{J}(v)}{\int_\Omega |x|^{-(N-4)}|V|\,|\nabla v - \frac{N-4}{2}\frac{x}{|x|^2}v|^2\,dx}, \tag{5.28}$$

are equal.

Proof Let c, $\mathbb{C}$ be the best constants in (5.27) and (5.28), respectively. For any $u \in C_0^\infty(\Omega)$ and $v = |x|^{\frac{N-4}{2}}u$, Lemma 2.3 implies that

$$\frac{\mathbb{I}[u]}{\int_\Omega |V|\,|\nabla u|^2\,dx} = \frac{\mathbb{J}(v)}{\int_\Omega |x|^{-(N-4)}|V|\,\tilde{v}^2\,dx},$$

hence $\mathsf{c} \geq \mathbb{C}$. Next we claim that $\mathsf{c} \leq \mathbb{C}$. Fix $\epsilon > 0$ and assume the functions $v_\epsilon \in C_0^\infty(\Omega)$, such that

$$\frac{\mathbb{J}(v_\epsilon)}{\int_\Omega |x|^{-(N-4)}|V|\,|\nabla v_\epsilon - \frac{N-4}{2}\frac{x}{|x|^2}v_\epsilon|^2\,dx} \leq \mathbb{C} + \epsilon.$$

Let $0 < a < \frac{N-4}{2}$. Lemma 5.1 implies that $u_{a,\epsilon} = |x|^{-a}v_\epsilon \in H_0^2(\Omega)$ providing that

$$\mathbb{C}_1 \leq \frac{\mathbb{I}[u_{a,\epsilon}]}{\int_\Omega |V|\,|\nabla u_{a,\epsilon}|^2\,dx} = \frac{\mathbb{J}_a(v_\epsilon)}{\int_\Omega |x|^{-2a}|V|\,|\nabla v_\epsilon - a\frac{x}{|x|^2}v_\epsilon|^2\,dx}, \tag{5.29}$$

where

$$\begin{aligned}
\mathbb{J}_a(v) &:= \int_\Omega |x|^{-2a}|\Delta v|^2\,dx - 4a(a+2)\int_\Omega |x|^{-2a-4}|x \cdot \nabla v|^2\,dx \\
&\quad + \left[2a(a+2) - \frac{N^2}{4}\right]\int_\Omega |x|^{-2a-2}|\nabla v|^2\,dx \\
&\quad + a(-N+a+4)\left[(a+2)(-N+a+2) + \frac{N^2}{4}\right]\int_\Omega |x|^{-2a-4}v^2\,dx.
\end{aligned}$$

Following similar arguments as in Lemma 5.3 we may prove that

$$a(-N+a+4)\left[(a+2)(-N+a+2) + \frac{N^2}{4}\right] \cdot \int_\Omega |x|^{-2a-4}v^2\,dx \to 0,$$

as $a \to \frac{N-4}{2}^-$, hence

$$\lim_{a \to \frac{N-4}{2}^-} \mathbb{J}_a(v) = \mathbb{J}(v), \tag{5.30}$$

for any $v \in C_0^\infty(\Omega/\{0\})$. Using now (5.30) and

$$\lim_{a \to \frac{N-4}{2}^-} \int_\Omega |x|^{-2a}|V|\,|\nabla v_\epsilon - a\frac{x}{|x|^2}v_\epsilon|^2\,dx = \int_\Omega |x|^{-(N-4)}|V|\,|\nabla v_\epsilon - \frac{N-4}{2}\frac{x}{|x|^2}v_\epsilon|^2\,dx$$

we obtain that

$$\mathsf{c} \leq \mathbb{C} + \epsilon,$$

for any fixed $\epsilon > 0$ and the proof is completed. ∎

By the same argument the Hardy-Sobolev inequality (1.10) takes the following form:

Lemma 5.13 *Let $D \geq \sup_{x \in \Omega} |x|$. Then, there exists $c > 0$, such that*

$$\mathbb{J}[v] \geq c \left(\int_\Omega |x|^{-\frac{N(N-4)}{N-2}} |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^{\frac{2N}{N-2}} X^{\frac{2N-2}{N-2}} \left(\frac{|x|}{D} \right) dx \right)^{\frac{N-2}{N}}, \tag{5.31}$$

for every $v \in W_0^{2,2}(\Omega, |x|^{-(N-4)})$.

Define the following quantity

$$\mathbb{Q}[v] := \frac{\mathbb{J}(v)}{\int_\Omega |x|^{-(N-4)} V \, |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^2 \, dx}$$

and set

$$\mathbb{B} := \inf_{\substack{v \in W_0^{2,2}(\Omega, |x|^{-(N-4)}), \\ \int_\Omega |x|^{-(N-4)} V |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^2 \, dx > 0}} \mathbb{Q}[v]. \tag{5.32}$$

By practically the same arguments as in Lemma 5.3 we have that

Lemma 5.14 *There holds: $\mathbb{B} = b$*

The local best constant of inequality (5.25) can be written as:

$$\mathbb{C}^0 := \lim_{r \downarrow 0} \mathbb{C}_r, \quad \mathbb{C}_r := \inf_{\substack{v \in C_0^\infty(B_r), \\ \int_{B_r} |x|^{-(N-4)} V |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^2 dx > 0}} \frac{\mathbb{J}(v)}{\int_{B_r} |x|^{-(N-4)} V \, |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^2 \, dx}. \tag{5.33}$$

If there is no $v \in C_0^\infty(B_r)$, such that $\int_{B_r} |x|^{-(N-4)} V |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^2 \, dx > 0$, for some $r > 0$, we set $\mathbb{C}_r = \infty$. Observe that $\mathbb{B} \leq \mathbb{C}^0$.

We introduce the space $\mathcal{V}$ to be defined as the closure of the $C_0^\infty(\Omega)$ functions with respect to the norm

$$\begin{aligned}
||u||_\mathcal{V}^2 &:= \int_\Omega |x|^{-(N-4)} |\Delta v|^2 \, dx + \int_\Omega |x|^{-N} |x \cdot \nabla v|^2 \, dx + \int_\Omega |x|^{-(N-2)} |\nabla v|^2 \, dx + \int_\Omega |x|^{-N} v^2 \, dx \\
&\quad + \int_\Omega |x|^{-(N-4)} |V| \left| \nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v \right|^2 dx.
\end{aligned} \tag{5.34}$$

It is clear that $\mathcal{V}$ is a Hilbert space with inner product

$$\begin{aligned}
<\phi, \psi>_\mathcal{V} &:= \int_\Omega |x|^{-(N-4)} \Delta\phi \, \Delta\psi \, dx + \int_\Omega |x|^{-N} (x \cdot \nabla\phi)(x \cdot \nabla\psi) \, dx + \int_\Omega |x|^{-(N-2)} \nabla\phi \cdot \nabla\psi \, dx \\
&\quad + \int_\Omega |x|^{-N} \phi \, \psi \, dx + \int_\Omega |x|^{-(N-4)} |V| (\nabla\phi - \frac{N-4}{2} \frac{x}{|x|^2} \phi)(\nabla\psi - \frac{N-4}{2} \frac{x}{|x|^2} \psi),
\end{aligned}$$

satisfying

$$||v||_W \leq ||v||_\mathcal{V} \leq c_0 ||v||_W, \quad \text{for any } v \in \mathcal{V}. \tag{5.35}$$

Theorem 5.15 *Let*

$$\mathbb{B} < \mathbb{C}^0. \tag{5.36}$$

Then $\mathbb{B}$ is achieved by some $v_0 \in W_0^{2,2}(\Omega, |x|^{-(N-4)})$.

Proof Let $\{v_k\} \subset W_0^{2,2}(\Omega, |x|^{-(N-4)})$ be a minimizing sequence for (5.25), such that

$$L(v_k) := \int_\Omega |x|^{-(N-4)} V \left|\nabla v_k - \frac{N-4}{2}\frac{x}{|x|^2} v_k\right|^2 dx = 1, \tag{5.37}$$

for every k. Hence $\mathbb{J}(v_k) \to B$. Since $\mathbb{J}(v_k)$ is bounded, from (2.23) and (2.18) we deduce that $\{v_k\}$ must be bounded too, in $W_0^{2,2}(\Omega, |x|^{-(N-4)})$. Therefore, there exists a subsequence, still denoted by $\{v_k\}$, such that

$$v_k \rightharpoonup v_0, \quad \text{in } W_0^{2,2}(\Omega, |x|^{-(N-4)})$$

and

$$v_k \to v_0, \quad \text{in } L^2(\Omega/B_\rho), \quad \text{for every } \rho > 0,$$

for some $v_0 \in W_0^{2,2}(\Omega, |x|^{-(N-4)})$. We set $w_k := v_k - v_0$. Then from (5.35) and (5.37) we have that

$$1 = L(w_k + v_0) = L(w_k) + L(v_0) + o(1). \tag{5.38}$$

Following now the same steps as in the proof of Proposition 5.7 we conclude that $L(v_0) \geq 1$, hence

$$\frac{\mathbb{J}}{L(v_0)} \geq \mathbb{B}.$$

This last inequality implies that $\mathbb{B}$ is attained by v_0, such that $L(v_0) = 1$ and the proof is completed. ∎

We next look for an improvement of inequality (5.25). That is, for an inequality of the form:

$$\int_\Omega |\Delta u|^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{|\nabla u|^2}{|x|^4} dx + \mathsf{b} \int_\Omega V|\nabla u|^2 dx + \mathsf{b}_1 \int_\Omega W|\nabla u|^2 dx, \quad u \in H_0^2(\Omega), \tag{5.39}$$

where V and W are both in $\mathbb{A}$. Assuming that (5.39) holds true, the best constant b_1, is clearly given by:

$$\mathsf{b}_1 = \inf_{\substack{u \in H_0^2(\Omega) \\ \int_\Omega W|\nabla u|^2 dx > 0}} \frac{\mathbb{I}[u] - b\int_\Omega V|\nabla u|^2 dx}{\int_\Omega W|\nabla u|^2 dx}. \tag{5.40}$$

By the same argument as in Lemma 5.12, the constant b_1 is also equal to:

$$\mathbb{B}_1 = \inf_{\substack{v \in W_0^{2,2}(\Omega, |x|^{-(N-4)}) \\ \int_\Omega |x|^{-(N-4)}W|\nabla v - \frac{N-4}{2}\frac{x}{|x|^2}v|^2 dx > 0}} \frac{\mathbb{J}[v] - b\int_\Omega |x|^{-(N-4)} V|\nabla v - \frac{N-4}{2}\frac{x}{|x|^2}v|^2 dx}{\int_\Omega |x|^{-(N-4)} W|\nabla v - \frac{N-4}{2}\frac{x}{|x|^2}v|^2 dx}. \tag{5.41}$$

Notice that by the properties of $\mathsf{b} = \mathbb{B}$ we always have that $\mathsf{b}_1 \geq 0$. Conversely, if one defines $\mathsf{b}_1 \geq 0$ by (5.41) it is immediate that inequality (5.39) holds true with b_1 being the best constant. But of course, for (5.39) to be an improvement of the original inequality, we need b_1 to be strictly positive. Our next result is a direct consequence of Proposition 5.15 and provides conditions under which the original inequality cannot be improved.

Theorem 5.16 *Suppose that $\mathbb{B} < \mathbb{C}^0$. Let V and W be both in $\mathbb{A}$. If ϕ is the minimizer of the quotient (5.32) and*

$$\int_\Omega |x|^{-(N-4)} W|\nabla\phi - \frac{N-4}{2}\frac{x}{|x|^2}\phi|^2 dx > 0,$$

then $\mathsf{b}_1 = 0$, that is, there is no further improvement of (5.25).

Proof By our assumptions, $v = \phi$ is an admissible function in (5.41). Moreover, for $v = \phi$ the numerator of (5.41) becomes zero. In view of the fact that $\mathsf{b}_1 \geq 0$, we conclude that $\mathsf{b}_1 = 0$. ∎

It follows in particular that if $W \geq 0$, we cannot improve (5.25). Thus, the following result has been proved.

Theorem 5.17 *Let $V \in \mathbb{A}$. If*

$$\mathbb{B} < \mathbb{C}^0,$$

then, we cannot improve (5.25) by adding a nonnegative potential $W \in \mathbb{A}$.

As a consequence of (1.10) and Theorem 5.17 we have:

Corollary 5.18 *Let $D > \sup_{x \in \Omega} |x|$. Suppose V is not everywhere nonpositive and such that*

$$\int_\Omega |V|^{\frac{N}{2}} X^{1-N}(|x|/D) \, dx < \infty.$$

Then, $V \in \mathbb{A}$ but there is no further improvement of (5.25) with a nonnegative $W \in \mathbb{A}$.

Proof Applying Holder's inequality we get:

$$\int_\Omega |x|^{-(N-4)} |V| |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^2 \, dx \leq \left(\int_\Omega |V|^{\frac{N}{2}} X^{1-N} \, dx \right)^{\frac{2}{N}} \cdot$$
$$\cdot \left(\int_\Omega |x|^{-\frac{N(N-4)}{N-2}} X^{\frac{2N-2}{N-2}} |\nabla v - \frac{N-4}{2} \frac{x}{|x|^2} v|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}}.$$

The first integral is bounded by our assumption, whereas the second integral is bounded from above by $C \, \mathbb{J}[v]$ (cf Lemma 5.13). Thus we proved that $V \in \mathbb{A}$. Using, as in the proof of Corollary 5.10, Holder's inequality in B_r and the definition of $\mathbb{C}_r$ (cf (5.33)) we easily get that $\mathbb{C}^0 = +\infty$. Thus, all conditions of Theorem 5.17 are satisfied and the result follows. ∎

6 PART II. THE POLYHARMONIC OPERATOR

In this part we prove some improved Hardy-Rellich inequalities involving the polyharmonic operator. More precisely, we give the proof of the Theorems 1.6 to 1.10 for which, we have to establish certain inequalities concerning (1.21) and (1.24).

6.1 The Inequality (1.21)

Lemma 6.1 *Suppose $N \geq 5$ and $0 \leq m < \frac{N-4}{2}$. For any $u \in C_0^\infty(\Omega)$, we set $v = |x|^a u$. Then, the following equality holds.*

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} \, dx = \int_\Omega |x|^{-2m-2a} |\Delta v|^2 \, dx - 4a(2m+2+a) \int_\Omega |x|^{-2a-4-2m} (x \cdot \nabla v)^2 \, dx$$

$$+ 2a(a+2+2m) \int_\Omega |x|^{-2a-2-2m} |\nabla v|^2 \, dx$$

$$+ \left(a^2(a+2-N)^2 - 2a(a+2-N)(m+1)(N-4-2m-2a) \right) \int_\Omega |x|^{-2a-4-2m} v^2 \, dx, \tag{6.1}$$

Lemma 6.2 *Suppose $N \geq 5$ and $0 \leq m < \frac{N-4}{2}$. For any $u \in C_0^\infty(\Omega)$, we set $v = |x|^{\frac{N-4-2m}{2}} u$. Then, the following equalities hold.*

i) $\quad \displaystyle\int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}}\, dx = \int_\Omega |x|^{-(N-2)}|\nabla v|^2\, dx + \left(\frac{N-4-2m}{2}\right)^2 \int_\Omega |x|^{-N}|v|^2\, dx,$

ii) $\quad \displaystyle\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}}\, dx - \left(\frac{(N+2m)(N-4-2m)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+4}}\, dx =$

$$\int_\Omega |x|^{-(N-4)}|\Delta v|^2\, dx - (N+2m)(N-4-2m)\int_\Omega |x|^{-N}(x\cdot\nabla v)^2\, dx$$

$$+\frac{(N+2m)(N-4-2m)}{2}\int_\Omega |x|^{-(N-2)}|\nabla v|^2\, dx,$$

Theorem 6.3 *Suppose $N \geq 5$ and $0 \leq m < \frac{N-4}{2}$. For any $u \in C_0^\infty(\Omega)$, we set $v = |x|^{\frac{N-4-2m}{2}} u$. Then, the following inequality holds.*

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}}\, dx - \left(\frac{(N+2m)(N-4-2m)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+4}}\, dx \geq A(N,m) \int_\Omega |x|^{-(N-2)}|\nabla v|^2\, dx, \qquad (6.2)$$

where

$$A(N,m) := \begin{cases} (N-1) + \frac{1}{2}(N+2m)(N-4-2m), & m > \frac{-2+\sqrt{N-1}}{2}, \\ 4(1+m)^2 + \frac{(N+2m)(N-4-2m)}{2}, & m \leq \frac{-2+\sqrt{N-1}}{2}, \end{cases}$$

Moreover, the constant $4(1+m)^2 + \frac{(N+2m)(N-4-2m)}{2}$ for $m < \frac{-2+\sqrt{N-1}}{2}$ is the best.

Proof We proceed by using Lemma 6.2 (ii) and decomposing v into spherical harmonics. The equalities (2.11)-(2.13) imply that is enough to prove that

$$\left[(k+1)^2 + (2k+N-1)(N-3) - \frac{1}{2}(N+2m)(N-4-2m) - A\right]\int_{\mathbb{R}^N} r^{2k-N+2}|\nabla g_k|^2\, dx$$

$$\geq \left[Ak(N-2) - \frac{k}{2}(N+2m)(N-4-2m)(2k+N-2)\right]\int_{\mathbb{R}^N} r^{2k-N}(g_k)^2\, dx. \qquad (6.3)$$

For $k = 0$ from (6.3) we obtain that

$$A \leq A_0 \equiv (N-2)^2 - \frac{1}{2}(N+2m)(N-4-2m),$$

while for $k \neq 0$ we obtain that

$$A \leq A_1 \equiv (N-1) + \frac{1}{2}(N+2m)(N-4-2m),$$

which corresponds for $k = 1$. Then, we conclude that A must be the minimum of A_0, A_1, or $A = A(N,m)$. Let $m < \frac{-2+\sqrt{N-1}}{2}$ and consider the minimizing sequences u^ϵ and v^ϵ

$$u^\epsilon := r^{-\frac{N-4}{2}+m+\epsilon} X_1^{\frac{-1+a_1}{2}}\phi(r), \qquad v^\epsilon := r^{\frac{N-4}{2}-m}u^\epsilon = r^\epsilon X_1^{\frac{-1+a_1}{2}}\phi(r),$$

in a similar way as in Section 4. Then, we have that

$$\frac{\int_\Omega \frac{|\Delta u^\epsilon|^2}{|x|^{2m}}\, dx - \left(\frac{(N+2m)(N-4-2m)}{4}\right)^2 \int_\Omega \frac{(u^\epsilon)^2}{|x|^{2m+4}}\, dx}{\int_\Omega |x|^{-(N-2)}|\nabla v^\epsilon|^2\, dx} \to 4(1+m)^2 + \frac{(N+2m)(N-4-2m)}{2},$$

as $\epsilon \to 0^+$ and $a_1 \to 0^+$. ∎

Observe that for $m = 0$ Proposition 6.3 implies that

$$A(N,0) = \left(4 + \frac{N(N-4)}{2}\right),$$

which is the result stated in Proposition 2.4.

Proof of Theorem 1.6 When $0 < m \leq \frac{-2+\sqrt{N-1}}{2}$, inequality (1.22) is an immediate consequence from Proposition 6.3 and Inequality (1.3) However, we will establish this for the whole range of $m \in [0, \frac{N-4}{4})$. Once more we do the change of variable of (2.11)-(2.13). Then the inequality will be true provided we will establish the following inequality

$$\left[(k+1)^2 + (2k+N-1)(N-3) - \frac{1}{2}(N+2m)(N-4-2m)\right] \int_{\mathbb{R}^N} r^{2k-N+2}|\nabla g_k|^2 \, dx$$

$$+ \frac{k}{2}(N+2m)(N-4-2m)(2k+N-2) \int_{\mathbb{R}^N} r^{2k-N}(g_k)^2 \sum_{i=1}^{k-1} X_1^2 \ldots X_k^2 \, dx$$

$$\geq \left[(1+m)^2 + \frac{(N+2m)(N-4-2m)}{8}\right] \int_{\mathbb{R}^N} r^{2k-N}(g_k)^2 \sum_{i=1}^{k-1} X_1^2 \ldots X_k^2 \, dx.$$

However, the worst case is for $k = 0$, but this follows from 1.5. To establish the best constants we will treat initially the case $m \leq \frac{-2+\sqrt{N-1}}{2}$. The proof of it follows the same lines as in Section 4. For this we fix small parameters $\epsilon, a_1, a_2, ..., a_k > 0$ and define

$$u(x) := w(x)\,\phi(|x|), \quad w(x) := |x|^{-\frac{N-4}{2}+m+\epsilon} X_1^{\frac{-1+a_1}{2}} X_2^{\frac{-1+a_2}{2}} \cdots X_k^{\frac{-1+a_k}{2}},$$

where $X_l = X_1(X_{l-1})$, $l = 2, ..., k$. and $\phi(r) \in C_0^\infty(B_1)$ is a smooth cutoff function, such that $0 \leq \phi \leq 1$, with $\phi \equiv 1$ in $B_{1/2}$. Following similar arguments as in Section 4 we may obtain that

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} \, dx - \left(\frac{(N+2m)(N-4-2m)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+4}} dx -$$

$$\left((1+m)^2 + \frac{(N+2m)(N-4-2m)}{8}\right) \sum_{i=1}^{k-1} \int_\Omega \frac{u^2}{|x|^{2m+4}} X_1^2 \cdots X_i^2 \, dx =$$

$$= c_N \int_0^1 r^{-1+2\epsilon} X_1^{-1+a_1} \cdots X_k^{-1+a_k} \left[\epsilon^2(2++2m+\epsilon)^2 - \frac{(N+2m)(N-4-2m)}{2}\epsilon(2+2m+\epsilon)\right.$$

$$+ 2(1+m+\epsilon)\left(-\frac{(N+2m)(N-4-2m)}{4} + \epsilon(2+2m+\epsilon)\right)\eta + (1+m+\epsilon)^2\eta^2$$

$$+ 2\left(-\frac{(N+2m)(N-4-2m)}{4} + \epsilon(2+2m+\epsilon)\right)(\tfrac{1}{4}\eta^2 + \tfrac{1}{2}B)$$

$$\left. - \left((1+m)^2 + \frac{(N+2m)(N-4-2m)}{8}\right)\sum_{i=1}^{k-1} X_1^2 \cdots X_i^2\right]\phi^2 \, dr + O(1),$$

Using now the identities (4.10), (4.11) and passing to the limit $\epsilon \to 0$, we conclude that

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} \, dx - \left(\frac{(N+2m)(N-4-2m)}{4}\right)^2 \int_\Omega \frac{u^2}{|x|^{2m+4}} dx$$

$$- \left((1+m)^2 + \frac{(N+2m)(N-4-2m)}{8}\right)\sum_{i=1}^{k-1} \int_\Omega \frac{u^2}{|x|^{2m+4}} X_1^2 \cdots X_i^2 \, dx =$$

$$= c_N((1+m)^2 + \frac{(N+2m)(N-4-2m)}{8})\left(A_k - \sum_{i=1}^k a_i A_i + \sum_{i=1}^{k-1}\sum_{j=i+1}^k (1-a_j)\Gamma_{ij}\right) + O(1).$$

However, we can pass to the limit $a_1 \downarrow 0, ... a_{k-1} \downarrow 0$ see (4.9), to conclude that the Rayleigh quotient now of (1.23) is smaller or equal than

$$\frac{\left((1+m)^2 + \frac{(N+2m)(N-4-2m)}{8}\right)\left(A_k - a_k A_k\right) + O(1)}{A_k} \to (1+m)^2 + \frac{(N+2m)(N-4-2m)}{8},$$

since $A_k \to \infty$, as $a_k \downarrow 0$. ∎

Proof of Theorem 1.9 Is an immediate consequence of the previous Theorem.

6.2   The Inequality (1.24)

In this section we consider Inequality (1.24). For our approach we consider decomposition into spherical harmonics, see Section 2. Let $u \in C_0^\infty(\Omega)$. Setting $u = u = \sum_{k=0}^\infty u_k := \sum_{k=0}^\infty f_k(r)\phi_k(\sigma)$, using equalities (2.5), (2.6) we have that

$$\int_{\mathbb{R}^N} \frac{|\Delta u_k|^2}{|x|^{2m}}\, dx = \int_{\mathbb{R}^N} r^{-2m}(f_k'')^2\, dx + \left[(N-1)(2m+1) + 2c_k\right]\int_{\mathbb{R}^N} r^{-2-2m}(f_k')^2\, dx$$

$$+ c_k\left[c_k + (N-4-2m)(2m+2)\right]\int_{\mathbb{R}^N} r^{-4-2m}(f_k)^2\, dx, \qquad (6.1)$$

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2m+2}}\, dx = \int_{\mathbb{R}^N} r^{-2-2m}(f_k')^2\, dx + c_k \int_{\mathbb{R}^N} r^{-4-2m}(f_k)^2\, dx. \qquad (6.2)$$

Theorem 6.6 *Suppose $N \geq 5$ and $0 \leq m < \frac{N-4}{2}$. Then, for any $u \in C_0^\infty(\Omega)$, the following inequality holds.*

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}}\, dx \geq a_{m,N} \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}}\, dx,$$

where $a_{m,N}$ is defined by:

$$a_{m,N} := \min_{k=0,1,2,...} \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + k(N+k-2)\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + k(N+k-2)}, \qquad (6.3)$$

In particular, we have

$$a_{m,N} = \left(\frac{N+2m}{2}\right)^2,$$

when $0 \leq m \leq \frac{-(N+4)+2\sqrt{N^2-N+1}}{6}$. Whereas, we have

$$a_{m,N} < \left(\frac{N+2m}{2}\right)^2,$$

when $\frac{-(N+4)+2\sqrt{N^2-N+1}}{6} < m < \frac{N-4}{2}$. Moreover, the minimum of (6.3) depends only on these k that satisfy

$$k \leq \left(\frac{\sqrt{3}}{3} - \frac{1}{2}\right)(N-2). \qquad (6.4)$$

and let $\bar{k}$ be the largest k of (6.4). In particular, for $N \leq 8$, and $\frac{-(N+4)+2\sqrt{N^2-N+1}}{6} < m < \frac{N-4}{2}$ we have

$$a_{m,N} = \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + N - 1\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + N - 1}$$

whereas, $8 < N$, the interval $(\frac{-(N+4)+2\sqrt{N^2-N+1}}{6}, \frac{N-4}{2})$ is been divided in $2\bar{k} - 1$ subintervals. For $k = 1, 2, ..., \bar{k}$

$$m_k^1 := \frac{2(N-5) - \sqrt{(N-2)^2 - 12k(k+N-2)}}{6},$$
$$m_k^2 := \frac{2(N-5) - \sqrt{(N-2)^2 + 12k(k+N-2)}}{6}.$$

When $m \in (\frac{-(N+4)+2\sqrt{N^2-N+1}}{6}, m_1^1] \cup [m_1^2, \frac{N-4}{2})$, then

$$a_{m,N} = \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + N - 1\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + N - 1}$$

For $2 \leq k \leq \bar{k} - 1$ and $m \in (m_k^1, m_{k+1}^1] \cup [m_{k+1}^2, m_k^2)$, then

$$a_{m,N} = \min\left\{ \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + k(N+k-2)\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + k(N+k-2)}, \right.$$
$$\left. \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + (k+1)(N+k-1)\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + (k+1)(N+k-1)} \right\}$$

For $m \in (m_{\bar{k}}^1, m_{\bar{k}}^2)$, then

$$a_{m,N} = \min\left\{ \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + \bar{k}(N+\bar{k}-2)\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + \bar{k}(N+\bar{k}-2)}, \right.$$
$$\left. \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + (\bar{k}+1)(N+\bar{k}-1)\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + (\bar{k}+1)(N+\bar{k}-1)} \right\}$$

Moreover, the constant $a_{m,N}$ in (1.24) is the best.

Proof Decomposing u into spherical harmonics. Using relations (6.1), (6.2), and the following Hardy inequality

$$\int_0^\infty r^{N-1-2m}(f_k'')^2\, dr \geq \left(\frac{N-2-2m}{2}\right)^2 \int_0^\infty r^{N-3-2m}(f_k')^2\, dr,$$

we obtain that

$$a_{m,N} \leq \frac{C_1 \frac{\int_0^\infty r^{N-3-2m}(f_k')^2\, dr}{\int_0^\infty r^{N-5-2m}(f_k)^2\, dr} + C_2}{\frac{\int_0^\infty r^{N-3-2m}(f_k')^2\, dr}{\int_0^\infty r^{N-5-2m}(f_k)^2\, dr} + c_k}$$

where $C_1 = \left[\left(\frac{N+2m}{2}\right)^2 + 2c_k\right]$, $C_2 = c_k\left[c_k - (N-3-2m)(N-4-2m) + (N-1)(N-4-2m)\right]$. However, since $C_2 - c_k C_1 \leq 0$, the real function

$$\omega(y) := \frac{C_1 y + C_2}{y + c_k} = \frac{C_1(y + c_k) + C_2 - c_k C_1}{y + c_k}$$

is increasing for positive y. Hence, from the Hardy inequality

$$\int_0^\infty r^{N-3-2m}(f_k')^2\, dr \geq \left(\frac{N-4-2m}{2}\right)^2 \int_0^\infty r^{N-5-2m}(f_k')^2\, dr,$$

we conclude that

$$a_{m,N} \leq A(k,N,m) := \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + c_k\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + c_k}.$$

We study the monotonicity of the function

$$f(x) = \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + x\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + x}, \quad x \geq 0.$$

It is clear that f admits a (possibly positive) minimum

$$x_0 := \frac{(N-4-2m)(-N+6m+8)}{4}.$$

Let $N \leq 8$, then $x_0 \geq 0$. In this case holds that $c_1 > x_0$, for $N = 5,6,7,8$ and $0 \leq m \leq \frac{N-4}{2}$, hence $a_{m,N} = \min\{A(0,N,m), A(1,N,m)\}$. Comparing $A(0,N,m)$, $A(1,N,m)$ i.e., $A(0,N,m) \leq A(1,N,m)$, we obtain that

$$(N+2m)(N-6m-8) + 4(N-1) \geq 0. \tag{6.5}$$

By simple calculations we may prove that

$$\text{if } 0 \leq m \leq m^* := \frac{-(N+4) + 2\sqrt{N^2 - N + 1}}{6} \quad \text{then } a_{m,N} = A(0,N,m),$$
$$\text{while for } m > m^*, \text{ we have that } a_{m,N} = A(1,N,m),$$

which in particular holds for every N. In the case where $m \leq \frac{N-8}{6}$, (clearly $N > 8$), we have that $x_0 \leq 0$ and f is increasing for all nonnegative x. Hence

$$a_{m,N} = A(0,N,m), \quad \text{for } N > 8 \text{ and } 0 \leq m \leq \frac{N-8}{6}.$$

Note that $\frac{N-8}{6} < m^*$, for every N. In the case where $0 < \frac{N-8}{6} \le m \le \frac{N-4}{2}$, the situation seems to be more complicated since $a_{m,N}$ also depends on some $k > 1$. Observe that $x_0 > 0$, which implies that f is decreasing for $x \in [0, x_0)$ and increasing for $x_0 < x$. In order to estimate the minimum $A(k, N, m)$, in terms of k, it suffices to find the relative position of x_0, as c_k varies; Let

$$\bar{k} := \max\{k \in \mathbb{N}, \text{ such that } c_k < x_0\},$$

(i.e. $c_{\bar{k}} < x_0 < c_{\bar{k}+1}$) then $a_{m,N} = \min\{A(\bar{k}, N, m), A(\bar{k}+1, N, m)\}$. However, $c_k < x_0$ implies that

$$12m^2 - 8(N-5)m + N^2 - 12N + 32 + 4c_k < 0.$$

Let $D := (N-2)^2 - 12c_k$, $m_k^1 := \frac{2(N-5)-\sqrt{D}}{6}$ and $m_k^2 := \frac{2(N-5)+\sqrt{D}}{6}$. Then, for every $k \in \mathbb{N}$, such that $D > 0$, (note that $D \ne 0$, for any k, N) there exist a whole interval of $m \in (\frac{N-8}{6}, \frac{N-4}{2})$, such that $a_{m,N} = A(k, N, m)$, as follows:

$$\text{if } m \in (m_k^1, m_{k+1}^1] \cup [m_{k+1}^2, m_k^2), \quad \text{then } a_{m,N} = \min\{A(k, N, m), A(k+1, N, m)\},$$

$$\text{while for } m \in (m_{\bar{k}}^1, m_{\bar{k}}^2), \quad \text{we have that } a_{m,N} = \min\{A(\bar{k}, N, m), A(\bar{k}+1, N, m)\}.$$

Having in mind that $m_0^1 = \frac{N-8}{6}$ and $m_0^2 = \frac{N-4}{2}$, we conclude that $a_{m,N}$ behaves in the way that the theorem states.

Finally, we prove that $a_{m,N}$ is the best constant. To this, let k be such that

$$a_{m,N} = \frac{\left(\frac{(N-4-2m)(N+2m)}{4} + k(N+k-2)\right)^2}{\left(\frac{N-4-2m}{2}\right)^2 + k(N+k-2)}.$$

We then set

$$u = |x|^{-\frac{N-4}{2}+m+\epsilon}\phi_k(\sigma)\phi(r),$$

where $\phi(r) \in C_0^\infty(B_1)$ is a smooth cutoff function, such that $0 \le \phi \le 1$, with $\phi \equiv 1$ in $B_{1/2}$ and $\phi_k(\sigma)$ is an eigenfunction of the Laplace-Beltrami operator with corresponding eigenvalue $c_k = k(N+k-2)$. Then we have that

$$\frac{1}{c_N} \int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} \, dx = \left(-\frac{(N+2m)(N-4-2m)}{4} - c_k + \epsilon(2+2m+\epsilon)\right)^2 \int_0^1 r^{-1+2\epsilon}\phi^2(r)dr + O(1),$$

$$\frac{1}{c_N} \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} \, dx = \left[\left(-\frac{N-4-2m}{2}+\epsilon\right)^2 + c_k\right] \int_0^1 r^{-1+2\epsilon}\phi^2(r)dr + O(1).$$

Letting now $\epsilon \downarrow 0$ we obtain the result. ∎

The requirement (6.4) implies that

$$k < \frac{2\sqrt{3}-3}{6}(N-2), \quad \text{or } k < 0.077\,(N-2). \tag{6.6}$$

From (6.6) it is clear that for $N < 15$ the quantity $a_{m,N}$ depends only on $k = 0, 1$. Thus, the case where $N = 9, ...14$, is similar to that of $N = 5, ..., 8$. However, there is a qualitative difference between these two cases, so we prefer to state Proposition 6.6 in this way. Observe that the above arguments still hold in the case of $m = 0$, see Proposition 2.4. As an example assume that $N = 30$. Then, from (6.1) we deduce

that $\bar{k} = 2$. We have also that $\frac{N-4}{2} = 13$, $m* \simeq 4.17$, $m_1^1 \simeq 4.85$, $m_2^1 = 7$, $m_2^2 \simeq 9.66$, $m_1^2 \simeq 11.81$. If we take for instance $m = 8$ we have that $x_0 = 65$, when $c_2 = 60$ and $c_3 = 93$. Actually, in this case we have that $A(k, N, m) = A(0, 30, 8) = 529$, $A(1, 30, 8) = 384$, $A(2, 30, 8) \simeq 360.29$, $A(3, 30, 8) \simeq 366.64$, hence $A(30, 8) = A(2, 30, 8)$.

Proof of Theorem 1.8 Let $V(x) = \sum_{i=1}^{\infty} X_1^2(\frac{|x|}{D}) X_2^2(\frac{|x|}{D}) \ldots X_i^2(\frac{|x|}{D})$. From relations (6.1), (6.2), inequality (1.26) is equivalent to

$$\int_{\Omega} r^{-2m}(f_k'')^2 \, dx - \left(\frac{N-2-2m}{2}\right)^2 \int_{\Omega} r^{-2-2m}(f_k')^2 \, dx - \frac{1}{4} \int_{\Omega} r^{-2-2m} V(x)(f_k')^2 \, dx$$

$$+2c_k \int_{\Omega} r^{-2-2m}(f_k')^2 dx + c_k \left[c_k + (N-4-2m)(2m+2) - \left(\frac{N+2m}{2}\right)^2 \right] \int_{\Omega} r^{-4-2m}(f_k)^2 dx$$

$$-\frac{c_k}{4} \int_{\Omega} r^{-4-2m} V(x)(f_k)^2 \, dx \geq 0. \qquad (6.7)$$

However, inequality (1.4) implies that

$$\int_{\Omega} r^{-2m}(f_k'')^2 \, dx - \left(\frac{N-2-2m}{2}\right)^2 \int_{\Omega} r^{-2-2m}(f_k')^2 \, dx - \frac{1}{4} \int_{\Omega} r^{-2-2m} V(x)(f_k')^2 \, dx \geq 0.$$

Hence, it suffices to prove that

$$2c_k \int_{\Omega} r^{-2-2m}(f_k')^2 \, dx + c_k \left[c_k + (N-4-2m)(2m+2) - \left(\frac{N+2m}{2}\right)^2 \right] \int_{\Omega} r^{-4-2m}(f_k)^2 \, dx$$

$$-\frac{1}{4}c_k \int_{\Omega} r^{-4-2m} V(x)(f_k)^2 \, dx \geq 0, \qquad (6.8)$$

or, since (6.8) holds for $k = 0$,

$$2 \int_{\Omega} r^{-2-2m}(f_k')^2 \, dx + \left[c_k + (N-4-2m)(2m+2) - \left(\frac{N+2m}{2}\right)^2 \right] \int_{\Omega} r^{-4-2m}(f_k)^2 \, dx$$

$$-\frac{1}{4} \int_{\Omega} r^{-4-2m} V(x)(f_k)^2 \, dx \geq 0, \qquad (6.9)$$

for any $k = 1, 2, \ldots$. Recalling again inequality (1.4), which gives

$$\int_{\Omega} r^{-2-2m}(f_k')^2 \, dx \geq \left(\frac{N-4-2m}{2}\right)^2 \int_{\Omega} r^{-4-2m}(f_k)^2 \, dx + \frac{1}{4} \int_{\Omega} r^{-4-2m} V(x)(f_k)^2 \, dx,$$

we obtain that (6.9) holds if

$$2\left(\frac{N-4-2m}{2}\right)^2 + c_k + (N-4-2m)(2m+2) - \left(\frac{N+2m}{2}\right)^2 \geq 0,$$

for any $k = 1, 2, \ldots$. However, this last inequality for $k = 1$ is equivalent to (6.5), which holds for $0 \leq m \leq \frac{-(N+4)+2\sqrt{N^2-N+1}}{6}$.

Assume now the minimizing sequences

$$u(x) := w(x)\,\phi(|x|), \quad w(x) := |x|^{-\frac{N-4}{2}+\epsilon} X_1^{\frac{-1+a_1}{2}} X_2^{\frac{-1+a_2}{2}} \cdots X_k^{\frac{-1+a_k}{2}},$$

introduced in Section 5 and using the same notation we have that

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} dx \;=\; \int_\Omega \frac{w^2}{|x|^{2m+4}} \left[\left(-\frac{(N+2m)(N-4-2m)}{4} + \epsilon(2+2m+\epsilon) \right)^2 + (1+\epsilon+m)^2\eta^2 \right.$$

$$+2(1+m+\epsilon)\left(-\frac{(N+2m)(N-4-2m)}{4} + \epsilon(2+2m+\epsilon) \right)\eta$$

$$\left. +2\left(-\frac{(N+2m)(N-4-2m)}{4} + \epsilon(2+2m+\epsilon) \right)(\tfrac{1}{4}\eta^2 + \tfrac{1}{2}B) \right] \cdots \phi^2\, dx + O(1),$$

and

$$\int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} X_1^2 \cdots X_i^2\, dx = \int_\Omega \frac{w^2}{|x|^{2m+4}} \left[\left(-\frac{N-4-2m}{2} + \epsilon \right)^2 + \left(-\frac{N-4-2m}{2} + \epsilon \right)\eta + \tfrac{1}{4}\eta^2 \right] \cdot$$

$$\cdot X_1^2 \cdots X_i^2\, \phi^2\, dx + O(1),$$

We now use identities (4.10), (4.11) and passing to the limit $\epsilon \to 0$, to conclude that

$$\int_\Omega \frac{|\Delta u|^2}{|x|^{2m}} dx - \left(\frac{N+2m}{2} \right)^2 \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} dx - \tfrac{1}{4}\sum_{i=1}^{k-1} \int_\Omega \frac{|\nabla u|^2}{|x|^{2m+2}} X_1^2 \cdots X_i^2\, dx =$$

$$= -\tfrac{1}{4}\left(\frac{N-4-2m}{2} \right)^2 c_N \int_0^1 r^{-1} X_1^{-1+a_1} \cdots X_i^{-1+a_k} \left[B + \sum_{i=1}^{k-1} X_1^2 \cdots X_i^2 \right] \phi^2\, dr + O(1),$$

$$= \tfrac{1}{4}\left(\frac{N-4-2m}{2} \right)^2 c_N A_k - \tfrac{1}{4}\left(\frac{N-4-2m}{2} \right)^2 c_N \left(\sum_{i=1}^{k} a_i A_i - \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} (1-a_j)\Gamma_{ij} \right) + O(1).$$

However, we can pass to the limit $a_1 \downarrow 0, ... a_{k-1} \downarrow 0$ see (4.9), to conclude that the Rayleigh quotient now of (1.27) is smaller or equal than

$$\frac{\tfrac{1}{4}\left(\frac{N-4-2m}{2} \right)^2 A_k - \tfrac{1}{4}\left(\frac{N-4-2m}{2} \right)^2 a_k A_k + O(1)}{\left(\frac{N-4-2m}{2} \right)^2 A_k + O(1)} \to \frac{1}{4},$$

since $A_k \to \infty$, as $a_k \downarrow 0$. ∎

Proof of Theorem 1.10 Is an immediate consequence of the previous Theorem.

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